

18.175: Lecture 25

Reflections and martingales

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Conditional expectation

Martingales

Arcsin law, other SRW stories

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Conditional expectation

- ▶ Say we're given a probability space $(\Omega, \mathcal{F}_0, P)$ and a σ -field $\mathcal{F} \subset \mathcal{F}_0$ and a random variable X measurable w.r.t. \mathcal{F}_0 , with $E|X| < \infty$. The **conditional expectation of X given \mathcal{F}** is a new random variable, which we can denote by $Y = E(X|\mathcal{F})$.
- ▶ We require that Y is \mathcal{F} measurable and that for all A in \mathcal{F} , we have $\int_A X dP = \int_A Y dP$.
- ▶ Any Y satisfying these properties is called a **version** of $E(X|\mathcal{F})$.
- ▶ Is it possible that there exists more than one version of $E(X|\mathcal{F})$ (which would mean that in some sense the conditional expectation is not canonically defined)?
- ▶ Is there some sense in which $E(X|\mathcal{F})$ always exists and is always uniquely defined (maybe up to set of measure zero)?

- ▶ **Claim:** Assuming $Y = E(X|\mathcal{F})$ as above, and $E|X| < \infty$, we have $E|Y| \leq E|X|$. In particular, Y is integrable.
- ▶ **Proof:** let $A = \{Y > 0\} \in \mathcal{F}$ and observe:
$$\int_A Y dP \int_A X dP \leq \int_A |X| dP.$$
 By similar argument,
$$\int_{A^c} -Y dP \leq \int_{A^c} |X| dP.$$
- ▶ **Uniqueness of Y :** Suppose Y' is \mathcal{F} -measurable and satisfies $\int_A Y' dP = \int_A X dP = \int_A Y dP$ for all $A \in \mathcal{F}$. Then consider the set $Y - Y' \geq \epsilon$. Integrating over that gives zero. Must hold for any ϵ . Conclude that $Y = Y'$ almost everywhere.

Radon-Nikodym theorem

- ▶ Let μ and ν be σ -finite measures on (Ω, \mathcal{F}) . Say $\nu \ll \mu$ (or ν is **absolutely continuous w.r.t.** μ) if $\mu(A) = 0$ implies $\nu(A) = 0$.
- ▶ Recall **Radon-Nikodym theorem**: If μ and ν are σ -finite measures on (Ω, \mathcal{F}) and ν is absolutely continuous w.r.t. μ , then there exists a measurable $f : \Omega \rightarrow [0, \infty)$ such that $\nu(A) = \int_A f d\mu$.
- ▶ Observe: this theorem implies existence of conditional expectation.

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Two big results

- ▶ **Optional stopping theorem:** Can't make money in expectation by timing sale of asset whose price is non-negative martingale.
- ▶ **Martingale convergence:** A non-negative martingale almost surely has a limit.

- ▶ **Wald's equation:** Let X_i be i.i.d. with $E|X_i| < \infty$. If N is a stopping time with $EN < \infty$ then $ES_N = EX_1EN$.
- ▶ **Wald's second equation:** Let X_i be i.i.d. with $E|X_i| = 0$ and $EX_i^2 = \sigma^2 < \infty$. If N is a stopping time with $EN < \infty$ then $ES_N = \sigma^2EN$.

- ▶ $S_0 = a \in \mathbb{Z}$ and at each time step S_j independently changes by ± 1 according to a fair coin toss. Fix $A \in \mathbb{Z}$ and let $N = \inf\{k : S_k \in \{0, A\}\}$. What is $\mathbb{E}S_N$?
- ▶ What is $\mathbb{E}N$?

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- ▶ How many walks from $(0, x)$ to (n, y) that don't cross the horizontal axis?
- ▶ Try counting walks that *do* cross by giving bijection to walks from $(0, -x)$ to (n, y) .

Ballot Theorem

- ▶ Suppose that in election candidate A gets α votes and B gets $\beta < \alpha$ votes. What's probability that A is ahead throughout the counting?
- ▶ Answer: $(\alpha - \beta)/(\alpha + \beta)$. Can be proved using reflection principle.

- ▶ Theorem for last hitting time.
- ▶ Theorem for amount of positive positive time.

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18.175 Theory of Probability

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