18.175: Lecture 2

Extension theorems: a tool for constructing measures

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Outline

Extension theorems

Distributions on $\mathbb{R}$

Extension theorems

Outline

Recall the dilemma

- Want, a priori, to define measure of *any* subsets of $[0, 1)$.
- Find that if we allow the axiom of choice and require measures to be countably additive (as we do) then we run into trouble. No valid translation invariant way to assign a finite measure to all subsets of $[0, 1)$.
- Could toss out the axiom of choice... but we don't want to. Instead we will only define measure for certain "measurable sets". We will construct a σ-algebra of measurable sets and let probability measure be function from σ-algebra to $[0, 1]$.
- Price to this decision: for the rest of our lives, whenever we talk about a measure on any space (a Euclidean space, a space of differentiable functions, a space of fractal curves embedded in a plane, etc.), we have to worry about what the σ-algebra might be.

Recall the dilemma

- On the other hand: always have to ensure that any measure we produce assigns actual number to every measurable set. A bigger σ-algebra means more sets whose measures have to be defined. So if we want to make it easy to construct measures, maybe it's a good thing if our σ-algebra doesn't have too many elements... unless it's easier to...
- Come to think of it, how do we define a measure anyway?
- If the σ-algebra is something like the Borel σ-algebra (smallest σ-algebra containing all open sets) it's a pretty big collection of sets. How do we go about producing a measure (*any* measure) that's defined for every set in this family?
- Answer: use extension theorems.

Recall definitions

- **Probability space** is triple $(\Omega, \mathcal{F}, P)$ where Ω is sample space, $\mathcal{F}$ is set of events (the σ-algebra) and $P : \mathcal{F} \to [0, 1]$ is the probability function.
- σ**-algebra** is collection of subsets closed under complementation and countable unions. Call $(\Omega, \mathcal{F})$ a measure space.
- **Measure** is function $\mu : \mathcal{F} \to \mathbb{R}$ satisfying $\mu(A) \geq \mu(\emptyset) = 0$ for all $A \in \mathcal{F}$ and countable additivity: $\mu(\cup_i A_i) = \sum_i \mu(A_i)$ for disjoint A_i.
- Measure μ is **probability measure** if $\mu(\Omega) = 1$.
- The **Borel** σ**-algebra** $\mathcal{B}$ on a topological space is the smallest σ-algebra containing all open sets.

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How do we produce measures on $\mathbb{R}$?

- Write $F(a) = P\big((-\infty, a]\big)$.
- **Theorem:** for each right continuous, non-decreasing function F, tending to 0 at $-\infty$ and to 1 at ∞, there is a unique measure defined on the Borel sets of $\mathbb{R}$ with $P((a, b]) = F(b) - F(a)$.
- If we're given such a function F, then we know how to compute the measure of any set of the form $(a, b]$.
- We would like to *extend* the measure defined for these subsets to a measure defined for the whole σ algebra generated by these subsets.
- Seems clear how to define measure of countable union of disjoint intervals of the form $(a, b]$ (just using countable additivity). But are we confident we can extend the definition to *all* Borel measurable sets in a consistent way?

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Algebras and semi-algebras

- **algebra**: collection $\mathcal{A}$ of sets closed under finite unions and complementation.
- **measure on algebra**: Have $\mu(A) \geq \mu(\emptyset) = 0$ for all A in $\mathcal{A}$, and for disjoint A_i with union in $\mathcal{A}$ we have $\mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$ (countable additivity).
- Measure μ on $\mathcal{A}$ is σ-**finite** if exists countable collection $A_n \in \mathcal{A}$ with $\mu(A_n) < \infty$ and $\cup A_n = \Omega$.
- **semi-algebra**: collection $\mathcal{S}$ of sets closed under intersection and such that $S \in \mathcal{S}$ implies that S^c is a finite disjoint union of sets in $\mathcal{S}$. (Example: empty set plus sets of form $(a_1, b_1] \times \ldots \times (a_d, b_d] \in \mathbb{R}^d$.)
- One lemma: If $\mathcal{S}$ is a semialgebra, then the set $\overline{S}$ of finite disjoint unions of sets in $\mathcal{S}$ is an algebra, called the **algebra generated by** $\mathcal{S}$.

π-systems and λ-systems

- Say collection of sets $\mathcal{P}$ is a π-system if closed under intersection.
- Say collection of sets $\mathcal{L}$ is a λ-system if
  - $\Omega \in \mathcal{L}$
  - If $A, B \in \mathcal{L}$ and $A \subset B$, then $B - A \in \mathcal{L}$.
  - If $A_n \in \mathcal{L}$ and $A_n \uparrow A$ then $A \in \mathcal{L}$.
- THEOREM: If $\mathcal{P}$ is a π-system and $\mathcal{L}$ is a λ-system that contains $\mathcal{P}$, then $\sigma(\mathcal{P}) \subset \mathcal{L}$, where $\sigma(\mathcal{A})$ denotes smallest σ-algebra containing $\mathcal{A}$.

Caratheéodory Extension Theorem

- **Theorem:** If μ is a σ-finite measure on an algebra $\mathcal{A}$ then μ has a unique extension to the σ algebra generated by $\mathcal{A}$.
- Detailed proof is somewhat involved, but let's take a look at it.
- We can use this extension theorem prove existence of a unique translation invariant measure (Lebesgue measure) on the Borel sets of $\mathbb{R}^d$ that assigns unit mass to a unit cube. (Borel σ-algebra $\mathcal{R}^d$ is the smallest one containing all open sets of $\mathbb{R}^d$. Given any space with a topology, we can define a σ-algebra this way.)

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