18.175: Lecture 15

Characteristic functions and central limit theorem

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Outline

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- Let X be a random variable.
- The **characteristic function** of X is defined by $\phi(t) = \phi_X(t) := E[e^{itX}]$.
- Recall that by definition $e^{it} = \cos(t) + i\sin(t)$.
- Characteristic function ϕ_X similar to moment generating function M_X.
- $\phi_{X+Y} = \phi_X \phi_Y$, just as $M_{X+Y} = M_X M_Y$, if X and Y are independent.
- And $\phi_{aX}(t) = \phi_X(at)$ just as $M_{aX}(t) = M_X(at)$.
- And if X has an mth moment then $E[X^m] = i^m \phi_X^{(m)}(0)$.
- Characteristic functions are well defined at all t for all random variables X.

Characteristic function properties

- $\phi(0) = 1$
- $\phi(-t) = \overline{\phi(t)}$
- $|\phi(t)| = |Ee^{itX}| \leq E|e^{itX}| = 1.$
- $|\phi(t+h) - \phi(t)| \leq E|e^{ihX} - 1|$, so $\phi(t)$ uniformly continuous on $(-\infty, \infty)$
- $Ee^{it(aX+b)} = e^{itb}\phi(at)$

Characteristic function examples

- **Coin:** If $P(X=1)=P(X=-1)=1/2$ then $\phi_X(t)=(e^{it}+e^{-it})/2=\cos t$.
- That's periodic. Do we always have periodicity if X is a random integer?
- **Poisson:** If X is Poisson with parameter λ then $\phi_X(t)=\sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k e^{itk}}{k!} = \exp(\lambda(e^{it}-1))$.
- Why does doubling λ amount to squaring ϕ_X?
- **Normal:** If X is standard normal, then $\phi_X(t)=e^{-t^2/2}$.
- Is ϕ_X always real when the law of X is symmetric about zero?
- **Exponential:** If X is standard exponential (density e^{-x} on $(0,\infty)$) then $\phi_X(t)=1/(1-it)$.
- **Bilateral exponential:** if $f_X(t)=e^{-|x|}/2$ on $\mathbb{R}$ then $\phi_X(t)=1/(1+t^2)$. Use linearity of $f_X \to \phi_X$.

Fourier inversion formula

- If $f:\mathbb{R}\to\mathbb{C}$ is in L^1, write $\hat{f}(t) := \int_{-\infty}^{\infty} f(x)e^{-itx}dx$.
- **Fourier inversion:** If f is nice: $f(x) = \frac{1}{2\pi}\int \hat{f}(t)e^{itx}dt$.
- Easy to check this when f is density function of a Gaussian. Use linearity of $f \to \hat{f}$ to extend to linear combinations of Gaussians, or to convolutions with Gaussians.
- Show $f \to \hat{f}$ is an isometry of Schwartz space (endowed with L^2 norm). Extend definition to L^2 completion.
- **Convolution theorem:** If

$$h(x) = (f*g)(x) = \int_{-\infty}^{\infty} f(y)g(x-y)dy,$$

then

$$\hat{h}(t) = \hat{f}(t)\hat{g}(t).$$

- **Possible application?**

$$\int 1_{[a,b]}(x)f(x)dx = (\widehat{1_{[a,b]}f})(0) = (\hat{f}*\widehat{1_{[a,b]}})(0) = \int \hat{f}(t)\widehat{1_{[a,b]}}(-t)dx.$$

Characteristic function inversion formula

- If the map $\mu_X \to \phi_X$ is linear, is the map $\phi \to \mu[a,b]$ (for some fixed $[a,b]$) a linear map? How do we recover $\mu[a,b]$ from ϕ?
- Say $\phi(t) = \int e^{itx}\mu(x)$.
- **Inversion theorem:**

$$\lim_{T\to\infty}(2\pi)^{-1}\int_{-T}^{T}\frac{e^{-ita}-e^{itb}}{it}\phi(t)dt = \mu(a,b) + \frac{1}{2}\mu(\{a,b\})$$

- **Main ideas of proof:** Write

$$I_T = \int \frac{e^{-ita}-e^{-itb}}{it}\phi(t)dt = \int_{-T}^{T}\int \frac{e^{-ita}-e^{-itb}}{it}e^{itx}\mu(x)dt.$$

- Observe that $\frac{e^{-ita}-e^{-itb}}{it} = \int_a^b e^{-ity}dy$ has modulus bounded by $b-a$.
- That means we can use Fubini to compute I_T.

Bochner's theorem

- Given any function ϕ and any points $x_1, \ldots, x_n$, we can consider the matrix with i, j entry given by $\phi(x_i - x_j)$. Call ϕ **positive definite** if this matrix is always positive semidefinite Hermitian.
- Bochner's theorem: a continuous function from $\mathbb{R}$ to $\mathbb{R}$ with $\phi(1) = 1$ is a characteristic function of a some probability measure on $\mathbb{R}$ if and only if it is positive definite.
- Positive definiteness kind of comes from fact that variances of random variables are non-negative.
- The set of all possible characteristic functions is a pretty nice set.

Continuity theorems

- **Lévy's continuity theorem:** if
$$\lim_{n\to\infty} \phi_{X_n}(t) = \phi_X(t)$$
for all t, then X_n converge in law to X.
- **Slightly stronger theorem:** If $\mu_n \implies \mu_\infty$ then $\phi_n(t) \to \phi_\infty(t)$ for all t. Conversely, if $\phi_n(t)$ converges to a limit that is continuous at 0, then the associated sequence of distributions μ_n is tight and converges weakly to measure μ with characteristic function ϕ.
- **Proof ideas:** First statement easy (since $X_n \implies X$ implies $Eg(X_n) \to Eg(X)$ for any bounded continuous g). To get second statement, first play around with Fubini and establish tightness of the μ_n. Then note that any subsequential limit of the μ_n must be equal to μ. Use this to argue that $\int f d\mu_n$ converges to $\int f d\mu$ for every bounded continuous f.

Moments, derivatives, CLT

- If $\int |x|^n \mu(x) < \infty$ then the characteristic function ϕ of μ has a continuous derivative of order n given by $\phi^{(n)}(t) = \int (ix)^n e^{itx} \mu(dx)$.
- Indeed, if $E|X|^2 < \infty$ and $EX = 0$ then $\phi(t) = 1 - t^2 E(X^2)/2o(t^2)$.
- This and the continuity theorem together imply the central limit theorem.
- **Theorem:** Let $X_1, X_2, \ldots$ by i.i.d. with $EX_i = \mu$, $\mathrm{Var}(X_i) = \sigma^2 \in (0, \infty)$. If $S_n = X_1 + \ldots + X_n$ then $(S_n - n\mu)/(\sigma n^{1/2})$ converges in law to a standard normal.

18.175 Theory of Probability
Spring 2014