18.175: Lecture 12

DeMoivre-Laplace and weak convergence

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Outline

DeMoivre-Laplace limit theorem

Weak convergence

Characteristic functions

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DeMoivre-Laplace limit theorem

- Let X_i be i.i.d. random variables. Write $S_n = \sum_{i=1}^n X_n$.
- Suppose each X_i is 1 with probability p and 0 with probability $q = 1 - p$.
- **DeMoivre-Laplace limit theorem:**

$$\lim_{n\to\infty} P\{a \le \frac{S_n - np}{\sqrt{npq}} \le b\} \to \Phi(b) - \Phi(a).$$

- Here $\Phi(b) - \Phi(a) = P\{a \le Z \le b\}$ when Z is a standard normal random variable.
- $\frac{S_n - np}{\sqrt{npq}}$ describes "number of standard deviations that S_n is above or below its mean".
- **Proof idea:** use binomial coefficients and Stirling's formula.
- Question: Does similar statement hold if X_i are i.i.d. from some other law?
- **Central limit theorem:** Yes, if they have finite variance.

Local $p = 1/2$ DeMoivre-Laplace limit theorem

- **Stirling:** $n! \sim n^n e^{-n} \sqrt{2\pi n}$ where $\sim$ means ratio tends to one.
- **Theorem:** If $2k/\sqrt{2n} \to x$ then $P(S_{2n} = 2k) \sim (\pi n)^{-1/2} e^{-x^2/2}$.

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Weak convergence

- Let X be random variable, X_n a sequence of random variables.
- Say X_n **converge in distribution** or **converge in law** to X if $\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$ at all $x \in \mathbb{R}$ at which F_X is continuous.
- Also say that the $F_n = F_{X_n}$ converge weakly to $F = F_X$.
- **Example:** X_i chosen from $\{-1, 1\}$ with i.i.d. fair coin tosses: then $n^{-1/2}\sum_{i=1}^n X_i$ converges in law to a normal random variable (mean zero, variance one) by Demoivre-Laplace.
- **Example:** If X_n is equal to $1/n$ a.s. then X_n converge weakly to an X equal to 0 a.s. Note that $\lim_{n\to\infty} F_n(0) \neq F(0)$ in this case.
- **Example:** If X_i are i.i.d. then the empirical distributions converge a.s. to law of X_1 (Glivenko-Cantelli).
- **Example:** Let X_n be the nth largest of $2n+1$ points chosen i.i.d. from fixed law.

Convergence results

- **Theorem:** If $F_n \to F_\infty$, then we can find corresponding random variables Y_n on a common measure space so that $Y_n \to Y_\infty$ almost surely.
- **Proof idea:** Take $\Omega = (0,1)$ and $Y_n = \sup\{y : F_n(y) < x\}$.
- **Theorem:** $X_n \implies X_\infty$ if and only if for every bounded continuous g we have $Eg(X_n) \to Eg(X_\infty)$.
- **Proof idea:** Define X_n on common sample space so converge a.s., use bounded convergence theorem.
- **Theorem:** Suppose g is measurable and its set of discontinuity points has μ_X measure zero. Then $X_n \implies X_\infty$ implies $g(X_n) \implies g(X)$.
- **Proof idea:** Define X_n on common sample space so converge a.s., use bounded convergence theorem.

Compactness

- **Theorem:** Every sequence F_n of distribution has subsequence converging to right continuous nondecreasing F so that $\lim F_{n(k)}(y) = F(y)$ at all continuity points of F.
- Limit may not be a distribution function.
- Need a "tightness" assumption to make that the case. Say μ_n are **tight** if for every ϵ we can find an M so that $\mu_n[-M, M] < \epsilon$ for all n. Define tightness analogously for corresponding real random variables or distributions functions.
- **Theorem:** Every subsequential limit of the F_n above is the distribution function of a probability measure if and only if the F_n are tight.

Total variation norm

- If we have two probability measures μ and ν we define the **total variation distance** between them is $||\mu - \nu|| := \sup_B |\mu(B) - \nu(B)|$.
- Intuitively, it two measures are close in the total variation sense, then (most of the time) a sample from one measure looks like a sample from the other.
- Convergence in total variation norm is much stronger than weak convergence.

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Characteristic functions

- Let X be a random variable.
- The **characteristic function** of X is defined by $\phi(t) = \phi_X(t) := E[e^{itX}]$. Like $M(t)$ except with i thrown in.
- Recall that by definition $e^{it} = \cos(t) + i \sin(t)$.
- Characteristic functions are similar to moment generating functions in some ways.
- For example, $\phi_{X+Y} = \phi_X \phi_Y$, just as $M_{X+Y} = M_X M_Y$, if X and Y are independent.
- And $\phi_{aX}(t) = \phi_X(at)$ just as $M_{aX}(t) = M_X(at)$.
- And if X has an mth moment then $E[X^m] = i^m \phi_X^{(m)}(0)$.
- But characteristic functions have an advantage: they are well defined at all t for all random variables X.

Continuity theorems

- **Lévy's continuity theorem:** if

$$\lim_{n\to\infty} \phi_{X_n}(t) = \phi_X(t)$$

  for all t, then X_n converge in law to X.
- By this theorem, we can prove the weak law of large numbers by showing $\lim_{n\to\infty} \phi_{A_n}(t) = \phi_\mu(t) = e^{it\mu}$ for all t. In the special case that $\mu = 0$, this amounts to showing $\lim_{n\to\infty} \phi_{A_n}(t) = 1$ for all t.
- **Moment generating analog:** if moment generating functions $M_{X_n}(t)$ are defined for all t and n and $\lim_{n\to\infty} M_{X_n}(t) = M_X(t)$ for all t, then X_n converge in law to X.