Lecture notes for class on Wednesday, April 22

1. The Strichartz inequality

The goal for the next couple lectures is to understand the Strichartz inequality for the Schrodinger equation. After that, we will start to study non-linear Schrodinger equations, and we will see that the Strichartz inequality plays an important role there.

We stated the Strichartz inequality a couple weeks ago. Let's recall it.

Theorem 1. *(Strichartz, 70's) Suppose that $u(x,t)$ obeys the Schrodinger equation on $\mathbb{R}^d \times \mathbb{R}$, $\partial_t u = i \triangle u$, with initial conditions $u(x,0) = u_0(x)$. Then u obeys the space-time L^s estimate*

$$\|u\|_{L^s_{x,t}} \lesssim \|u_0\|_{L^2},$$

where $s = \frac{2(d+2)}{d}$.

The exponent s is the only exponent which is consistent with the scaling $u_\lambda(x,t) = u(x/\lambda, t/\lambda^2)$.

Let us recall what the solution to the Schrodinger equation is like. Taking the Fourier transform of the equation, we see that

$$\partial_t \hat{u}(\omega, t) = i(2\pi i)^2 |\omega|^2 \hat{u}(\omega, t).$$

Therefore,

$$\hat{u}(\omega, t) = e^{i(2\pi i)^2 |\omega|^2 t} \hat{u}_0(\omega).$$

Therefore $u(x,t)$ is given by the inverse Fourier transform of the right hand side, which we write as $e^{it\triangle} u_0$:

$$u(x,t) = e^{it\triangle} u_0(x) := \left(e^{i(2\pi i)^2 |\omega|^2 t} \hat{u}_0(\omega) \right)^{\vee} (x).$$

The notation $e^{it\triangle}$ is suggested because applying the Laplacian in physical space is equivalent to multiplying in Fourier space by $(2\pi i)^2 |\omega|^2$. Another intuition for this notation is that when we write down that $e^{it\triangle} u_0$ satisfies the Schrodinger equation, we write

$$\partial_t (e^{it\triangle} u_0) = i \triangle (e^{it\triangle} u_0).$$

By the way, the solution is defined for all $t \in \mathbb{R}$, not just $t > 0$, and the same formulas make sense for negative t.

So far, we have learned two estimates about solutions to the Schrodinger equation. We write these estimates in terms of the notation $e^{it\triangle}u_0$. First, the L^2 norm of a solution is preserved in time:

$$\|e^{it\triangle}u_0\|_{L^2_x} = \|u_0\|_{L^2_x}.$$

Second, solutions of the Schrodinger equation obey an L^∞ decay estimate:

$$\|e^{it\triangle}u_0\|_{L^\infty_x} \lesssim |t|^{-d/2}\|u_0\|_{L^1_x}.$$

These two facts play a crucial role in proving the Strichartz inequality, but it is quite tricky to put them together.

It is probably helpful to keep in mind a couple examples. Suppose that $w(x,t)$ solves the Schrodinger equation with initial data w_0 equal to a smooth bump function on the unit ball $B(1) \subset \mathbb{R}^d$. For times t with $|t| \gtrsim 1$, the solution $w(x,t)$ behaves roughly as follows: $|w(x,t)| \sim t^{-d/2}$ on a ball of radius $|t|$, and decays rapidly for $|x| \gg |t|$. We check that $\|w(x,t)\|^2_{L^2_x} \sim |B^d(t)| \cdot (t^{-d/2})^2 \sim 1$. This example shows that the decay estimate is sharp.

Here is a slightly more interesting example. Fix some large $T > 0$, and define

$$v_0(x) = w(x, -T).$$

We have $v(x,t) = w(x, t-T)$, so we can easily understand v. In particular $e^{iT\triangle}v_0 = w_0$. The decay estimate is also sharp for v_0 and time $t = T$. Note that $\|v_0\|_{L^1_x} \sim |B^d(T)| \cdot T^{-d/2} \sim T^{d/2}$. The decay estimate gives that

$$\|w_0\|_{L^\infty_x} = \|e^{iT\triangle}v_0\|_{L^\infty_x} \lesssim T^{-d/2}\|v_0\|_{L^1} \lesssim 1.$$

Since $\|w_0\|_{L^\infty_x} \sim 1$, the decay estimate used must have been essentially sharp. By the way, note that $\|e^{iT\triangle}v_0\|_{L^\infty}$ is much larger than $\|v_0\|_{L^\infty}$. The function v_0 is called a focusing example. Even though we use the word "decay estimate", we have to understand that this can happen – it is an important phenomenon in studying the Schrodinger equation.

We have two estimates – the conservation of L^2 and the decay estimate. Now that we have proven the interpolation theorem, we can interpolate between these two estimates.

Proposition 2. *For any $0 \le \theta \le 1$, define p by*

$$\frac{1}{p} = (1-\theta) \cdot \frac{1}{2},$$

and let p' be the dual exponent. Then we have the following inequality.

$$\|e^{it\triangle}u_0\|_{L^p_x} \lesssim t^{-\frac{d}{2}\cdot\theta}\|u_0\|_{L^{p'}_x}.$$

This inequality is essentially sharp for all θ. In fact, both examples above are sharp: if we take w_0 and any $|t| \geq 1$, or if we take v_0 and time $t = T$, then the L^p estimate in the Proposition is sharp up to a constant factor.

This Proposition seems like a good step towards estimating the L^p norm of the solution on space and time. If we apply this estimate in the simplest way, the following happens.

$$\|e^{it\triangle}u_0\|^p_{L^p_{x,t}} = \int_{\mathbb{R}} \|e^{it\triangle}u_0\|^p_{L^p_x} dt \leq \int_{\mathbb{R}} t^{-\frac{d}{2}\cdot\theta p}\|u_0\|^p_{L^{p'}_x} dt.$$

The integral in t never converges. Also we are particularly interested in $\|u_0\|_{L^2_x}$, which forces $p = p' = 2$, and we don't get a global estimate. For $t > 0$, there is no fixed time estimate of the form

$$\|e^{it\triangle}u_0\|_{L^p_x} \leq C(t)\|u_0\|_{L^2_x}.$$

The reason is that $e^{it\triangle}$ is an isometric bijection from L^2_x to itself. So, given any function w with $\|w\|_{L^2_x} = 1$, we can find u_0 with $e^{it\triangle}u_0 = w$ and $\|u_0\|_{L^2_x} = 1$. We can also find an explicit counterexample by rescaling the focusing example v_0 above. The Strichartz inequality says that

$$\int_{\mathbb{R}} \|e^{it\triangle}u_0\|^s_{L^s_x} dt \lesssim \|u_0\|^s_{L^2_x},$$

so although we can't bound the integrand at any single value of t, we can still bound the integral on the left-hand side. The L^2-mass of u may focus for a small set of times t, but the Strichartz inequality shows that it cannot remain focused over a large set of times.

In some sense, we will prove the Strichartz inequality using the L^2 estimate and the decay estimate, but in a sort-of round about way. This argument involves introducing some more characters.

2. The inhomogeneous Schrodinger equation

There are several variations of the Strichartz inequality, and Theorem 1 is actually not the easiest. We start by widening our perspective. We consider the inhomogeneous Schrodinger equation

$$\partial_t u = i\triangle u + F.$$

Here u and F are both functions of x and t. We will write $F_t(x)$ for $F(x,t)$. Similarly, we will write $u_t(x)$ for $u(x,t)$.

A solution to the inhomogenenous Schrodinger equation is given in the following proposition.

Proposition 3. *(Duhamel formula) If $F \in C^{\infty}_{comp}(\mathbb{R}^d \times \mathbb{R})$, then the following function u solves the inhomogeneous Schrodinger equation:*

$$u_t = \int_{-\infty}^{t} e^{i(t-s)\triangle} F_s ds.$$

Moreover, the function $u(x,t)$ vanishes at all times t "before" the support of F.

Proof. The last claim is easy to check. Suppose that F is supported on $\mathbb{R}^d \times [T_1, T_2]$. If $t < T_1$, then $F_s = 0$ for all $s \in [-\infty, t]$, and so $u_t = 0$.

Recall that $e^{it\triangle} u_0$ solves the Schrodinger equation:

$$\partial_t \left(e^{it\triangle} u_0 \right) = i\triangle \left(e^{it\triangle} u_0 \right).$$

So taking the time derivative of u_t, we get

$$\partial_t u_t = e^{i(t-s)\triangle} F_s|_{s=t} + \int_{-\infty}^{t} \partial_t \left(e^{i(t-s)\triangle} F_s \right) ds =$$

$$= F_t + \int_{-\infty}^{t} i\triangle \left(e^{i(t-s)\triangle} F_s \right) ds = F_t + i\triangle u_t.$$

□

There is another Strichartz inequality that relates the size of F and the size of u. This is a cousin of the first Strichartz inequality we stated. It is a little bit easier to prove, but we will see later that it implies Theorem 1. This theorem is the heart of the matter.

Theorem 4. *(Also Strichartz) Suppose that u obeys the inhomogeneous Strichartz equation $\partial_t u = i\triangle u + F$, and that u vanishes at times before the support of F. Let s be the Strichartz exponent $s = \frac{2(d+2)}{d}$ as above, and let s' be its dual exponent. Then*

$$\|u\|_{L^s_{x,t}} \lesssim \|F\|_{L^{s'}_{x,t}}.$$

Proof. We will use the Duhamel formula, and the L^p estimates in Proposition 2. For any p, we have

$$\|u\|^p_{L^p_{x,t}} = \int_{\mathbb{R}} \|u_t\|^p_{L^p_x} dx.$$

By Duhamel's formula and Minkowski's inequality,

$$\|u_t\|_{L^p_x} = \left\| \int_{-\infty}^{t} e^{i(t-s)\triangle} F_s \right\|_{L^p_x} \leq \int_{-\infty}^{t} \|e^{i(t-s)\triangle} F_s\|_{L^p_x}.$$

As in Propositon 2, let's suppose that $\frac{1}{p} = (1-\theta) \cdot \frac{1}{2}$. Applying Proposition 2, we see that

$$\|u_t\|_{L^p_x} \lesssim \int_{-\infty}^{t} (t-s)^{-\frac{d}{2}\theta} \|F_s\|_{L^{p'}_x}. \tag{1}$$

The right-hand side is a convolution which is a little hard to see with all the notation. We let $g(s) = \|F_s\|_{L^{p'}_x}$, we let $h(s) = \|u_s\|_{L^p_x}$, and we let $\alpha = \frac{d}{2}\theta$. Then the last equation gives

$$h(t) \leq g * |t|^{-\alpha}(t). \tag{2}$$

Note that $\|h\|_p = \|u\|_{L^p_{x,t}}$ and $\|g\|_{p'} = \|F\|_{L^{p'}_{x,t}}$.

By Hardy-Littlewood-Sobolev, and equation (2), we know that $\|h\|_r \lesssim \|g\|_q$ provided that

$$\frac{1}{r} + 1 = \frac{1}{q} + \alpha.$$

In particular, $\|h\|_p \lesssim \|g\|_{p'}$ as long as

$$\frac{1}{p} + 1 = \frac{p-1}{p} + \frac{d}{2} \cdot \theta.$$

When we plug in $p = s$ and find the corresponding θ, this equation is satisfied, and so we get $\|u\|_{L^s_{x,t}} \lesssim \|F\|_{L^{s'}_{x,t}}$ as desired. We do the computation with s and θ here in the notes for completeness, although I'm not sure if it's illuminating enough to include in the lecture.

Recall that p and θ are related by $\frac{1}{p} = (1-\theta)\frac{1}{2}$, which yields $\theta = 1 - \frac{2}{p} = \frac{p-2}{p}$. Plugging for θ in the last equation, we get

$$\frac{1}{p} + 1 = \frac{p-1}{p} + \frac{d(p-2)}{2p}.$$

Multiplying through by $2p$, we get

$$2 + 2p = 2(p-1) + d(p-2).$$

$$4 = d(p-2).$$

$$p = \frac{4}{d} + 2 = \frac{2(d+2)}{d} = s.$$

□

Next class, we'll discuss this proof more, and we'll see how Theorem 1 follows from Theorem 4.

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