

# 18.156 Differential Analysis II

## Lectures 1-2

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Throughout these notes, we will be working typically over  $\mathbb{R}^n$ , and we will use coordinates  $x_1, \dots, x_n$  when necessary. The convention will be that the Laplacian on  $\mathbb{R}^n$  with the standard Euclidean metric is given by the analyst's sign convention  $\Delta = \sum_{i=1}^n \left( \frac{\partial}{\partial x_i} \right)^2$ .

#### 1.1 Theory of Harmonic Functions

There are three main properties of harmonic functions which we will quickly prove, assuming the results to be mostly review. These are the Maximum Principle, the Mean Value Property, and regularity. A main focus of the class will be on how these properties transfer over to more general operators other than the Laplacian.

##### 1.1.1 Maximum Principle

**Lemma 1.1.** *If  $\Omega \subset \mathbb{R}^n$  is open with  $\bar{\Omega}$  compact and  $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$  with  $\Delta u > 0$ , then*

$$\max_{\bar{\Omega}} u = \max_{\partial\Omega} u.$$

*Proof.* Since  $\bar{\Omega}$  is compact,  $u$  attains its maximum, say at  $x \in \bar{\Omega}$ . Suppose by way of contradiction that  $x \in \Omega$ . Then since  $x$  is a maximum, we must have that  $\partial_i u(x) = 0$  and  $\partial_i^2 u(x) \leq 0$  for all  $i$ . Summing over  $i$  yields  $\Delta u \leq 0$ , and we establish a contradiction.  $\square$

**Lemma 1.2** (Maximum Principle for Harmonic Functions). *Suppose  $\Omega$  and  $u$  are as in the previous lemma but with  $\Delta u = 0$  (i.e.  $u$  is harmonic). Then the same conclusion follows, i.e.*

$$\max_{\bar{\Omega}} u = \max_{\partial\Omega} u.$$

*Proof.* Consider for  $\epsilon > 0$  the function  $u_\epsilon(x) = u(x) + \epsilon|x|^2$ . Then one calculates  $(\Delta u_\epsilon)(x) = (\Delta u)(x) + 2n\epsilon = 2n\epsilon > 0$ . Hence the previous lemma applies to show  $u_\epsilon$  attains its maximum on the boundary, and taking the limit as  $\epsilon \rightarrow 0$  yields the result.  $\square$

**Corollary 1.3.** *If  $\Delta u = \Delta v$  with  $u, v \in C^2(\Omega) \cap C^0(\bar{\Omega})$  and  $u|_{\partial\Omega} = v|_{\partial\Omega}$ , then  $u = v$  on all of  $\Omega$ .*

*Proof.* We have that  $\Delta(u - v) = 0$  with  $u - v = 0$  on  $\partial\Omega$ , so by the Maximum Principle,  $u \leq v$  on  $\Omega$ . Symmetrically,  $v \leq u$  on  $\Omega$ . Hence,  $u = v$  on  $\Omega$ .  $\square$

##### 1.1.2 Mean Value Property

From now on, we will denote by  $B(x, r)$  or  $B_r(x)$  the ball of radius  $r$  centered at  $x$ , and by  $S(x, r)$  or  $S_r(x)$  the sphere of radius  $r$  centered at  $x$ . If there is no  $x$  in the notation, it will usually be clear in context. Many times, we will take  $x$  to be the origin, but not always, so do pay attention.

**Theorem 1.4** (Mean Value Property). *If  $u \in C^2(\mathbb{R}^n)$  with  $\Delta u = 0$ , then for all  $r > 0$ , we have  $u(x) = \int_{B(x,r)} u(y) dy$ .*

*Proof.* We have

$$\begin{aligned}
\frac{d}{dr} \int_{S_r(x)} u &= \int_{S_r} \partial_{nor} u \\
&= \frac{1}{|S_r|} \int_{\partial B_r} \partial_{nor} u \\
&= \frac{1}{|S_r|} \int_{B_r} \operatorname{div} \nabla u \\
&= \frac{1}{|S_r|} \int_{B_r} \Delta u = 0
\end{aligned}$$

where we have used the Divergence Theorem and the fact that  $\Delta = \operatorname{div} \nabla$ . Hence,  $\int_{S_r} u$  is constant, and letting  $r \rightarrow 0$ , we see from the continuity of  $u$  that this constant is  $u(x)$ .  $\square$

This proof is somehow a bit mysterious, and doesn't really tell us what's going on under the hood. We present a second proof which takes into account the role of symmetries. We will only do this for the case of  $n = 2$  and with  $x = 0$ , but this can be easily generalized.

*Alternate proof of MVP,  $n = 2$ .* For  $\theta \in SO(2) \simeq \mathbb{R}/2\pi\mathbb{Z}$  we have a corresponding rotation  $R_\theta: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ , which, as an isometry, preserves the Laplacian. Hence, if we define

$$(\theta \cdot u)(y) := u(R_\theta(y)),$$

then  $R_\theta u$  is still harmonic. Next set

$$(Au)(y) := \frac{1}{2\pi} \int_0^{2\pi} (\theta \cdot u)(y) d\theta.$$

We see immediately by taking the Laplacian, which commutes with the integral, that  $Au$  is also harmonic. Now by definition,  $(Au)(y) = \int_{S_{|y|}} u$  is the function we have to prove is constant. But by construction,  $Au$  is constant on spheres centered at the origin, and so the uniqueness of Corollary 1.3 yields that  $Au$  is constant on the interior of any disk.  $\square$

### 1.1.3 Regularity

**Theorem 1.5.** *If  $u \in C^2(\mathbb{R}^n)$  with  $\Delta u = 0$  on  $B_1$ , then there is some constant  $C_n$  depending only on the dimension  $n$  such that*

$$\max_{B_{1/2}} |\nabla u| \leq C_n \cdot \max_{B_1} |u|.$$

*Proof.* Suppose  $x \in B_{1/2}$ . Then by the mean value property and the fact that  $B_{1/2}(x) \subseteq B_1$ , we have

$$u(x) = \frac{1}{|B_{1/2}|} \int_{B_{1/2}(x)} u(y) dy.$$

We wish to take the derivative with respect to  $x$ , which is only found in the domain of integration, not the integrand. One sees that

$$\partial_i u(x) = \frac{1}{|B_{1/2}|} \int_{S_{1/2}(x)} \langle nor, \partial_i \rangle u(y) dy,$$

whereby the notation  $\langle nor, \partial_i \rangle$  means the dot product of the vector  $\partial_i$  with the normal vector at the point  $y \in S(x, 1/2)$ . These coefficients don't actually depend on  $y$ , and we see immediately that

$$|\partial_i u(x)| \leq C_n \max |u|$$

for some constant  $C_n$  depending only on  $n$ .  $\square$

**Corollary 1.6.** *If  $u \in C^2(B_1)$  and  $\Delta u = 0$  on  $B_1$ , then there is some constant  $C(k, n)$  depending only on  $k$  and  $n$  such that*

$$\max_{B_{1/2}} |\partial^k u| \leq C(k, n) \max_{B_1} |u|$$

**Remark 1.7.** *If a function is  $C^2$  and harmonic, then it is  $C^\infty$ . (Even further, Weyl's lemma tells us that harmonic distributions are  $C^\infty$ .)*

## 1.2 More General Situations

We will deal with more general operators than just  $\Delta$ .

- **Variable coefficient operators** are of the form  $Lu(x) = \sum_{i,j=1}^n a_{ij}(x) \partial_i \partial_j u(x)$  where for each  $x$ ,  $[a_{ij}(x)]$  is a symmetric matrix (as we can always assume). We may want to restrict to cases where  $[a_{ij}(x)]$  has eigenvalues all strictly positive. The Laplacian  $\Delta$  is an example of this where the  $a_{ij}$  are constants.
- **Non-linear operators** might look something like  $\Delta u = \pm u^3$ .
- **Quasi-linear operators** are of the form  $\sum a_{ij}(u, \partial_k u) \partial_i \partial_j u(x) = 0$ . These look difficult to study, but they come up in geometry such as in the minimal surface equation, so they are worth attention.

One might wonder whether the Maximum Principle, Mean Value Property, or Regularity results that harmonic functions enjoy are shared by variable coefficient operators. We will be discussing in what ways this is the case. A big first theorem for us to prove is Schauder's.

Recall the idea of Hölder-regularity. For  $0 < \alpha \leq 1$ , we have the semi-norms

$$[f]_{C^\alpha} := \sup_{x,y} \frac{|f(x) - f(y)|}{|x - y|^\alpha},$$

and corresponding norms  $\|f\|_{C^\alpha} := [f]_{C^\alpha} + \|f\|_{C^0}$ , where these are defined on some space of functions. Then

**Theorem 1.8** (Schauder Interior Estimate). *If  $Lu$  is a variable coefficient operator as above with  $0 < \lambda \leq \text{eig}[a_{ij}] \leq \Lambda$  and  $\|a_{ij}\|_{C^\alpha(B_1)} \leq B$  for some  $0 < \alpha < 1$ , and in addition there is some  $u \in C^2(B_1)$  such that  $Lu = 0$  on  $B_1$ , then*

$$\|u\|_{C^{2,\alpha}(B_{1/2})} \leq C(\lambda, \Lambda, B, n, \alpha) \cdot \|u\|_{C^0(B_1)}$$

where  $\|u\|_{C^{2,\alpha}} = \|u\|_{C^2} + [\partial^2 u]_{C^\alpha}$ .

How might we prove something like this? Well, WSLOG, we can assume that  $a_{ij}(x_0) = \delta_{ij}$ , and then look at some small ball around  $x_0$  on which we will have  $L = \Delta + \text{small order terms}$ . So  $Lu = 0$  means that  $u$  is almost harmonic. We will come back to the problem of putting estimates on almost harmonic functions.

As a toy example, one might ask a related problem. Suppose  $u: B_1 \rightarrow \mathbb{R}$  is  $C^2$  with  $|\Delta u(x)| \leq 10^{-6}$  and  $|u(x)| \leq 1$  on  $B_1$ . Then can we say  $|\nabla u(0)| \leq 10^6$ ?

## Lecture 2: 6 February 2015

### 2.1 Almost Harmonic Functions

**Proposition 2.1.** *If  $u \in C_{cpt}^3(\mathbb{R}^n)$ , then*

$$\|\partial^2 u\|_{L^2} = \|\Delta u\|_{L^2}.$$

*Proof.* This follows from two applications of integration by parts:

$$\begin{aligned}
\|\partial^2 u\|_{L^2}^2 &= \sum_{i,j} \int_{\mathbb{R}^n} (\partial_i \partial_j u)(\partial_i \partial_j u) \\
&= - \sum_{i,j} \int_{\mathbb{R}^n} (\partial_i^2 \partial_j u)(\partial_j u) \\
&= \sum_{i,j} \int_{\mathbb{R}^n} (\partial_i^2 u)(\partial_j^2 u) \\
&= \int_{\mathbb{R}^n} \left( \sum_i \partial_i^2 u \right) \left( \sum_j \partial_j^2 u \right) \\
&= \|\Delta u\|_{L^2}^2
\end{aligned}$$

□

**Proposition 2.2.** *If  $|a_{ij}(x) - \delta_{ij}| < \epsilon(n)$  for all  $x \in \mathbb{R}^n$  and all  $1 \leq i, j \leq n$ , where  $\epsilon(n)$  is some small enough constant depending only on  $n$  (we will see we can take  $\epsilon \leq (2n)^{-1}$ ), and if  $L = \sum_{i,j} a_{ij}(x) \partial_i \partial_j$ , then for any  $u \in C_{cpt}^3(\mathbb{R}^n)$ , we have*

$$\|\partial^2 u\|_{L^2} \leq 2\|Lu\|_{L^2}.$$

*Proof.* We have

$$\begin{aligned}
\|\partial^2 u\|_{L^2}^2 &= \int |\partial^2 u|^2 \\
&= \int |\Delta u|^2 \\
&= \int |Lu + (\Delta - L)u|^2 \\
&\leq \int |Lu|^2 + 2 \int |Lu| |(\Delta - L)u| + \int |(\Delta - L)u|^2
\end{aligned}$$

Now, the coefficients of  $\Delta - L$  are  $a_{ij} - \delta_{ij}$ , which are bounded in absolute value by  $\epsilon$ , so continuing the chain of inequalities, we have

$$\|\partial^2 u\|_{L^2}^2 \leq \|Lu\|_{L^2}^2 + 2\epsilon n \int |Lu| |\partial^2 u| + \epsilon^2 n^2 \|\partial^2 u\|_{L^2}^2.$$

Applying the Cauchy-Schwarz inequality to the middle term, this yields

$$\|\partial^2 u\|_{L^2}^2 \leq 2\|Lu\|_{L^2}^2 + 2\epsilon^2 n^2 \|\partial^2 u\|_{L^2}^2.$$

If we choose  $\epsilon \leq (2n)^{-1}$ , then rearranging yields

$$\frac{1}{2} \|\partial^2 u\|_{L^2}^2 \leq (1 - 2\epsilon^2 n^2) \|\partial^2 u\|_{L^2}^2 \leq 2\|Lu\|_{L^2}^2.$$

□

**Proposition 2.3.** *If  $L$  is as in Proposition 2.2, and  $u \in C^3(B_1)$  with  $Lu = 0$  on  $B_1$ , then*

$$\|\partial^2 u\|_{L^2(B_{1/2})} \leq C(n) \|\partial u + |u|\|_{L^2(B_1)}$$

for some  $C(n)$  depending only on  $n$ .

*Proof.* The idea is to apply Proposition 2.2 to a localized version of  $u$ . Namely, take  $\eta$  a smooth bump function supported on  $B_1$  with  $\eta = 1$  on  $B_{1/2}$ . Then

$$\begin{aligned} \|\partial^2 u\|_{L^2(B_1)}^2 &= \int_{B_{1/2}} |\partial^2 u|^2 \\ &= \int_{B_{1/2}} |\partial^2(\eta u)|^2 \\ &\leq \int_{B_1} |\partial^2(\eta u)|^2 \\ &\leq 4 \int_{B_1} |L(\eta u)|^2 \\ &\leq 4 \int_{B_1} \eta^2 |Lu|^2 + C(n) \int |\nabla u| |\nabla \eta| + C(n) \int |\partial^2 \eta| |u| \end{aligned}$$

The first term is zero, and for the other two, an application of Cauchy-Schwarz separates the  $u$  and  $\eta$  terms in into different integrals so that we get in total

$$\|\partial^2 u\|_{L^2(B_1)}^2 \leq C(n) D(\eta) (\|\partial u\|_{L^2(B_1)}^2 + \|u\|_{L^2(B_1)}^2)$$

where  $D$  only depends on  $\eta$ , a choice of which only depended on  $n$ . □

Now, we showed that  $\|\partial^2 u\|_{L^2} \leq \|\Delta u\|_{L^2}$  (in fact we proved equality). This is both somewhat surprising and also very powerful. One might ask if there are similar bounds in other norms. We provide four examples, along with their answers:

1. The inequality **does not** hold in the  $C^0$ -norm, which we will see in Lecture 4.
2. The inequality **does** hold in the  $L^p$ -norm, and this is the Calderon-Zygmund inequality from the 1950s.
3. The inequality **does not** hold in the  $C^1$ -norm.
4. The inequality **does** hold in the  $C^\alpha$ -norm. This is Korn's inequality from the early 1900s. This will be proved in Lecture 5.

Our goal over the next week will be to work especially towards Korn's inequality. Essentially, we have the analogy that Proposition 2.1 is to Proposition 2.3 as Korn's inequality is to Schauder's inequality, which we stated last lecture. Our true main goal for this section of the class is Schauder's inequality.

## 2.2 Solving $\Delta u = f$ (a story from physics)

On our way to Korn's inequality, we want to understand how we can write explicit estimates on the second partial derivatives of  $u$  in terms of  $\Delta u$ . Here, the story begins with some classical physics theory. We will make the mathematics a bit more precise next lecture.

The theory starts with gravity. In Newtonian mechanics, if there is a point mass at the origin, then the force felt by any particle in  $\mathbb{R}^3$  is given by

$$F(x) = \frac{-x}{|x|^3}.$$

It didn't take long for people to notice that  $F = -\nabla \Gamma$ , where  $\Gamma = \frac{1}{|x|}$  is the potential energy, called the Newton potential. Further,  $\operatorname{div} F = \Delta \Gamma = 0$ .

**Theorem 2.4** (Newton). *The gravitational field generated by a uniform mass on the ball  $B_r$  is the same outside the ball as the gravitational field of the corresponding point mass at the center.*

*Proof.* For a point  $z \in B_r$  and  $x \in \mathbb{R}^3 \setminus B_r$ , set  $F_z(x) = -\frac{x-z}{|x-z|^3}$ . What the theorem is stating is that

$$\int_{B_r} F_z(x) dz = |B_r| F_0(x), \quad \forall x \notin B_r.$$

**Lemma 2.5.** *If  $\Omega$  is a precompact open set in  $\mathbb{R}^3$  with smooth boundary, and  $z \in \Omega$ , then*

$$\int_{\partial\Omega} F_z(x) \cdot nor = +4\pi.$$

(If  $z \notin \Omega$ , the same proof will show that this integral is just 0.)

*Proof.* Set  $U = \Omega \setminus B_r(z)$  where  $r$  is small enough so that the entire ball lies inside  $\Omega$ . Then the divergence theorem shows that

$$0 = \int_U \operatorname{div} F_z = \int_{\partial U} F_z \cdot nor = \int_{\partial\Omega} F_z \cdot nor - \int_{\partial B_r(z)} F_z \cdot nor.$$

But this last term is explicitly calculated to be  $4\pi$ . □

So now set  $V(x) = \int_{B_r} F_z(x) dz$ . By symmetry,  $V(x)$  is always parallel to  $x$ . It suffices to check that

$$\int_{\partial B(R)} V \cdot nor = \int_{\partial B(R)} F_0 \cdot nor |B_r|.$$

The RHS is easily seen to be  $4\pi|B_r|$  (by the calculation we stated at the end of the lemma), while the LHS is

$$\int_{\partial B(R)} \left( \int_{B_r} F_z(x) dz \right) \cdot nor \, dx = \int_{B_r} \left( \int_{\partial B(R)} F_z(x) \cdot nor \, dx \right) dz = \int_{B_r} 4\pi dz = 4\pi|B_r|,$$

where we have used the lemma. □

Now suppose that  $f(z)$  is some mass distribution. Then we have

- The gravity field at  $x$  is  $F(x) = \int f(z) F_z(x) dx$ .
- The potential energy at  $x$  is  $u(x) = \int f(z) \Gamma(x - z) dz$ .

Laplace noticed that  $\Delta u = f$ . So in this way, we have an explicit solution to  $\Delta u = f$  for fixed  $f$ .

Further, if  $u \in C^2$ , we have that for a domain  $\Omega$  as in Newton's Theorem,

$$\int_{\Omega} \Delta u = \int_{\partial\Omega} \nabla u \cdot nor.$$

We can take this instead to be our definition of the Laplacian in the sense of distributions. Doing this, note that from the Lemma, we had

$$\int_{\Omega} \nabla \Gamma \cdot nor = \begin{cases} 4\pi, & 0 \in \Omega \\ 0, & 0 \notin \Omega \end{cases}$$

so that in the distributional sense,

$$\Delta \Gamma = 4\pi \delta_0.$$

If we believe this, then it is not hard to show that

$$u(x) = \int f(z) \Gamma(x - z) dz \implies \Delta u = f.$$

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