## 9. Fourier inversion

It is shown above that the Fourier transform satisfies the identity

$$\varphi(0) = (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{\varphi}(\xi)\, d\xi \ \forall\ \varphi \in \mathcal{S}(\mathbb{R}^n)\,. \tag{9.1}$$

If $y \in \mathbb{R}^n$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ set $\psi(x) = \varphi(x+y)$. The translation-invariance of Lebesgue measure shows that

$$\begin{aligned} \hat{\psi}(\xi) &= \int e^{-ix\cdot\xi}\varphi(x+y)\, dx \\ &= e^{iy\cdot\xi}\hat{\varphi}(\xi)\,. \end{aligned}$$

Applied to $\psi$ the inversion formula (9.1) becomes

$$\begin{aligned} \varphi(y) = \psi(0) &= (2\pi)^{-n} \int \hat{\psi}(\xi)\, d\xi \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{iy\cdot\xi}\hat{\varphi}(\xi)\, d\xi\,. \end{aligned} \tag{9.2}$$

**Theorem 9.1.** *Fourier transform $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is an isomorphism with inverse*

$$\mathcal{G} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)\,,\ \mathcal{G}\psi(y) = (2\pi)^{-n} \int e^{iy\cdot\xi}\psi(\xi)\, d\xi\,. \tag{9.3}$$

*Proof.* The identity (9.2) shows that $\mathcal{F}$ is $1-1$, i.e., injective, since we can remove $\varphi$ from $\hat{\varphi}$. Moreover,

$$\mathcal{G}\psi(y) = (2\pi)^{-n}\mathcal{F}\psi(-y) \tag{9.4}$$

So $\mathcal{G}$ is also a continuous linear map, $\mathcal{G} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$. Indeed the argument above shows that $\mathcal{G} \circ \mathcal{F} = Id$ and the same argument, with some changes of sign, shows that $\mathcal{F} \cdot \mathcal{G} = Id$. Thus $F$ and $\mathcal{G}$ are isomorphisms.

□

**Lemma 9.2.** *For all $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$, Paseval's identity holds:*

$$\int_{\mathbb{R}^n} \varphi\overline{\psi}\, dx = (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{\varphi}\overline{\hat{\psi}}\, d\xi\,. \tag{9.5}$$

*Proof.* Using the inversion formula on $\varphi$,

$$\int \varphi\overline{\psi}\,dx = (2\pi)^{-n}\int \left(e^{ix\cdot\xi}\hat{\varphi}(\xi)\,d\xi\right)\overline{\psi}(x)\,dx$$
$$= (2\pi)^{-n}\int \hat{\varphi}(\xi)\overline{\int e^{-ix\cdot\xi}\psi(x)\,dx}\,d\xi$$
$$= (2\pi)^{-n}\int \hat{\varphi}(\xi)\overline{\hat{\varphi}}(\xi)\,d\xi\,.$$

Here the integrals are absolutely convergent, justifying the exchange of orders.

□

**Proposition 9.3.** *Fourier transform extends to an isomorphism*

(9.6) $$\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)\,.$$

*Proof.* Setting $\varphi = \psi$ in (9.5) shows that

(9.7) $$\|\mathcal{F}\varphi\|_{L^2} = (2\pi)^{n/2}\|\varphi\|_{L^2}\,.$$

In particular this proves, given the known density of $\mathcal{S}(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$, that $\mathcal{F}$ is an isomorphism, with inverse $\mathcal{G}$, as in (9.6).

□

For any $m \in \mathbb{R}$

$$\langle x\rangle^m L^2(\mathbb{R}^n) = \left\{u \in \mathcal{S}'(\mathbb{R}^n)\,;\,\langle x\rangle^{-m}\hat{u} \in L^2(\mathbb{R}^n)\right\}$$

is a well-defined subspace. We define the *Sobolev spaces* on $\mathbb{R}^n$ by, for $m \geq 0$

(9.8) $$H^m(\mathbb{R}^n) = \left\{u \in L^2(\mathbb{R}^n)\,;\,\hat{u} = \mathcal{F}u \in \langle\xi\rangle^{-m}L^2(\mathbb{R}^n)\right\}\,.$$

$$\text{Thus } H^m(\mathbb{R}^n) \subset H^{m'}(\mathbb{R}^n) \text{ if } m \geq m'\,,\ H^0(\mathbb{R}^n) = L^2(\mathbb{R}^n)\,.$$

**Lemma 9.4.** *If $m \in \mathbb{N}$ is an integer, then*

(9.9) $$u \in H^m(\mathbb{R}^n) \Leftrightarrow D^\alpha u \in L^2(\mathbb{R}^n)\ \forall\ |\alpha| \leq m\,.$$

*Proof.* By definition, $u \in H^m(\mathbb{R}^n)$ implies that $\langle\xi\rangle^{-m}\hat{u} \in L^2(\mathbb{R}^n)$. Since $\widehat{D^\alpha u} = \xi^\alpha\hat{u}$ this certainly implies that $D^\alpha u \in L^2(\mathbb{R}^n)$ for $|\alpha| \leq m$. Conversely if $D^\alpha u \in L^2(\mathbb{R}^n)$ for all $|\alpha| \leq m$ then $\xi^\alpha\hat{u} \in L^2(\mathbb{R}^n)$ for all $|\alpha| \leq m$ and since

$$\langle\xi\rangle^m \leq C_m\sum_{|\alpha|\leq m}|\xi^\alpha|\,.$$

this in turn implies that $\langle\xi\rangle^m\hat{u} \in L^2(\mathbb{R}^n)$.

□

Now that we have considered the Fourier transform of Schwartz test functions we can use the usual method, of duality, to extend it to tempered distributions. If we set $\eta = \overline{\hat{\psi}}$ then $\hat{\psi} = \overline{\eta}$ and $\psi = \mathcal{G}\hat{\psi} = \mathcal{G}\overline{\eta}$ so

$$\overline{\psi}(x) = (2\pi)^{-n} \int e^{-ix\cdot\xi}\overline{\hat{\psi}}(\xi)\, d\xi$$
$$= (2\pi)^{-n} \int e^{-ix\cdot\xi}\eta(\xi)\, d\xi = (2\pi)^{-n}\hat{\eta}(x).$$

Substituting in (9.5) we find that

$$\int \varphi\hat{\eta}\, dx = \int \hat{\varphi}\eta\, d\xi\,.$$

Now, recalling how we embed $\mathcal{S}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$ we see that

$$u_{\hat{\varphi}}(\eta) = u_{\varphi}(\hat{\eta})\ \forall\ \eta \in \mathcal{S}(\mathbb{R}^n)\,. \tag{9.10}$$

**Definition 9.5.** *If* $u \in \mathcal{S}'(\mathbb{R}^n)$ *we* define *its Fourier transform by*

$$\hat{u}(\varphi) = u(\hat{\varphi})\ \forall\ \varphi \in \mathcal{S}(\mathbb{R}^n)\,. \tag{9.11}$$

As a composite map, $\hat{u} = u \cdot \mathcal{F}$, with each term continuous, $\hat{u}$ is continuous, i.e., $\hat{u} \in \mathcal{S}'(\mathbb{R}^n)$.

**Proposition 9.6.** *The definition* (9.7) *gives an isomorphism*

$$\mathcal{F} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)\,,\ \mathcal{F}u = \hat{u}$$

*satisfying the identities*

$$\widehat{D^\alpha u} = \xi^\alpha u\,,\ \widehat{x^\alpha u} = (-1)^{|\alpha|}D^\alpha\hat{u}\,. \tag{9.12}$$

*Proof.* Since $\hat{u} = u \circ \mathcal{F}$ and $\mathcal{G}$ is the 2-sided inverse of $\mathcal{F}$,

$$u = \hat{u} \circ \mathcal{G} \tag{9.13}$$

gives the inverse to $\mathcal{F} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$, showing it to be an isomorphism. The identities (9.12) follow from their counterparts on $\mathcal{S}(\mathbb{R}^n)$:

$$\widehat{D^\alpha u}(\varphi) = D^\alpha u(\hat{\varphi}) = u((-1)^{|\alpha|}D^\alpha\hat{\varphi})$$
$$= u(\widehat{\xi^\alpha\varphi}) = \hat{u}(\xi^\alpha\varphi) = \xi^\alpha\hat{u}(\varphi)\ \forall\ \varphi \in \mathcal{S}(\mathbb{R}^n)\,.$$

□

We can also define Sobolev spaces of *negative* order:

$$H^m(\mathbb{R}^n) = \left\{u \in \mathcal{S}'(\mathbb{R}^n)\,;\ \hat{u} \in \langle\xi\rangle^{-m}L^2(\mathbb{R}^n)\right\}\,. \tag{9.14}$$

**Proposition 9.7.** *If $m \le 0$ is an integer then $u \in H^m(\mathbb{R}^n)$ if and only if it can be written in the form*

$$u = \sum_{|\alpha| \le -m} D^\alpha v_\alpha \,,\; v_\alpha \in L^2(\mathbb{R}^n)\,. \tag{9.15}$$

*Proof.* If $u \in \mathcal{S}'(\mathbb{R}^n)$ is of the form (9.15) then

$$\hat{u} = \sum_{|\alpha| \le -m} \xi^\alpha \hat{v}_\alpha \text{ with } \hat{v}\alpha \in L^2(\mathbb{R}^n)\,. \tag{9.16}$$

Thus $\langle\xi\rangle^m \hat{u} = \sum_{|\alpha| \le -m} \xi^\alpha \langle\xi\rangle^m \hat{v}_\alpha$. Since all the factors $\xi^\alpha\langle\xi\rangle^m$ are bounded, each term here is in $L^2(\mathbb{R}^n)$, so $\langle\xi\rangle^m \hat{u} \in L^2(\mathbb{R}^n)$ which is the definition, $u \in \langle\xi\rangle^{-m} L^2(\mathbb{R}^n)$.

Conversely, suppose $u \in H^m(\mathbb{R}^n)$, i.e., $\langle\xi\rangle^m \hat{u} \in L^2(\mathbb{R}^n)$. The function

$$\left(\sum_{|\alpha| \le -m} |\xi^\alpha|\right) \cdot \langle\xi\rangle^m \in L^2(\mathbb{R}^n)\ (m < 0)$$

is bounded below by a positive constant. Thus

$$v = \left(\sum_{|\alpha| \le -m} |\xi^\alpha|\right)^{-1} \hat{u} \in L^2(\mathbb{R}^n)\,.$$

Each of the functions $\hat{v}_\alpha = \operatorname{sgn}(\xi^\alpha)\hat{v} \in L^2(\mathbb{R}^n)$ so the identity (9.16), and hence (9.15), follows with these choices.

□

**Proposition 9.8.** *Each of the Sobolev spaces $H^m(\mathbb{R}^n)$ is a Hilbert space with the norm and inner product*

$$\|u\|_{H^m} = \left(\int_{\mathbb{R}^n} |\hat{u}(\xi)|^2 \langle\xi\rangle^{2m}\, d\xi\right)^{1/2}, \tag{9.17}$$
$$\langle u, v\rangle = \int_{\mathbb{R}^n} \hat{u}(\xi)\overline{\hat{v}(\xi)}\langle\xi\rangle^{2m}\, d\xi\,.$$

*The Schwartz space $\mathcal{S}(\mathbb{R}^n) \hookrightarrow H^m(\mathbb{R}^n)$ is dense for each $m$ and the pairing*

$$H^m(\mathbb{R}^n) \times H^{-m}(\mathbb{R}^n) \ni (u, u') \longmapsto \tag{9.18}$$
$$((u, u')) = \int_{\mathbb{R}^n} \hat{u'}(\xi)\hat{u'}(\cdot\xi)\, d\xi \in \mathbb{C}$$

*gives an identification $(H^m(\mathbb{R}^n))' = H^{-m}(\mathbb{R}^n)$.*

*Proof.* The Hilbert space property follows essentially directly from the definition (9.14) since $\langle\xi\rangle^{-m}L^2(\mathbb{R}^n)$ is a Hilbert space with the norm (9.17). Similarly the density of $\mathcal{S}$ in $H^m(\mathbb{R}^n)$ follows, since $\mathcal{S}(\mathbb{R}^n)$ dense in $L^2(\mathbb{R}^n)$ (Problem L11.P3) implies $\langle\xi\rangle^{-m}\mathcal{S}(\mathbb{R}^n) = \mathcal{S}(\mathbb{R}^n)$ is dense in $\langle\xi\rangle^{-m}L^2(\mathbb{R}^n)$ and so, since $\mathcal{F}$ is an isomorphism in $\mathcal{S}(\mathbb{R}^n)$, $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^m(\mathbb{R}^n)$.

Finally observe that the pairing in (9.18) makes sense, since $\langle\xi\rangle^{-m}\hat{u}(\xi)$, $\langle\xi\rangle^{m}\hat{u'}(\xi) \in L^2(\mathbb{R}^n)$ implies

$$\hat{u}(\xi))\hat{u'}(-\xi) \in L^1(\mathbb{R}^n)\,.$$

Furthermore, by the self-duality of $L^2(\mathbb{R}^n)$ each continuous linear functional

$$U : H^m(\mathbb{R}^n) \to \mathbb{C}\,,\; U(u) \leq C\|u\|_{H^m}$$

can be written uniquely in the form

$$U(u) = ((u, u')) \text{ for some } u' \in H^{-m}(\mathbb{R}^n)\,.$$

□

Notice that if $u, u' \in \mathcal{S}(\mathbb{R}^n)$ then

$$((u, u')) = \int_{\mathbb{R}^n} u(x)u'(x)\,dx\,.$$

This is always how we "pair" functions — it is the natural pairing on $L^2(\mathbb{R}^n)$. Thus in (9.18) what we have shown is that this pairing on test function

$$\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \ni (u, u') \longmapsto ((u, u')) = \int_{\mathbb{R}^n} u(x)u'(x)\,dx$$

extends by *continuity* to $H^m(\mathbb{R}^n) \times H^{-m}(\mathbb{R}^n)$ (for each fixed $m$) when it identifies $H^{-m}(\mathbb{R}^n)$ as the dual of $H^m(\mathbb{R}^n)$. This was our 'picture' at the beginning.

For $m > 0$ the spaces $H^m(\mathbb{R}^n)$ represents elements of $L^2(\mathbb{R}^n)$ that have "$m$" derivatives in $L^2(\mathbb{R}^n)$. For $m < 0$ the elements are ?? of "up to $-m$" derivatives of $L^2$ functions. For integers this is precisely ??.