8. Convolution and density

We have defined an inclusion map

$$\mathcal{S}(\mathbb{R}^n) \ni \varphi \longmapsto u_\varphi \in \mathcal{S}'(\mathbb{R}^n),\ u_\varphi(\psi) = \int_{\mathbb{R}^n} \varphi(x)\psi(x)\,dx\ \forall\ \psi \in \mathcal{S}(\mathbb{R}^n). \tag{8.1}$$

This allows us to 'think of' $\mathcal{S}(\mathbb{R}^n)$ as a subspace of $\mathcal{S}'(\mathbb{R}^n)$; that is we habitually identify u_φ with φ. We can do this because we know (8.1) to be injective. We can extend the map (8.1) to include bigger spaces

$$\begin{aligned} \mathcal{C}^0_0(\mathbb{R}^n) \ni \varphi &\longmapsto u_\varphi \in \mathcal{S}'(\mathbb{R}^n) \\ L^p(\mathbb{R}^n) \ni \varphi &\longmapsto u_\varphi \in \mathcal{S}'(\mathbb{R}^n) \\ M(\mathbb{R}^n) \ni \mu &\longmapsto u_\mu \in \mathcal{S}'(\mathbb{R}^n) \\ u_\mu(\psi) &= \int_{\mathbb{R}^n} \psi\,d\mu\,, \end{aligned} \tag{8.2}$$

but we need to know that these maps are injective before we can forget about them.

We can see this using *convolution*. This is a sort of 'product' of functions. To begin with, suppose $v \in \mathcal{C}^0_0(\mathbb{R}^n)$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$. We define a new function by 'averaging v with respect to ψ:'

$$v * \psi(x) = \int_{\mathbb{R}^n} v(x-y)\psi(y)\,dy\,. \tag{8.3}$$

The integral converges by dominated convergence, namely $\psi(y)$ is integrable and v is bounded,

$$|v(x-y)\psi(y)| \le \|v\|_{\mathcal{C}^0_0}\ |\psi(y)|\ .$$

We can use the same sort of estimates to show that $v*\psi$ is continuous. Fix $x \in \mathbb{R}^n$,

$$\begin{aligned} &v * \psi(x+x') - v*\psi(x) \\ &\qquad = \int (v(x+x'-y) - v(x-y))\psi(y)\,dy\,. \end{aligned} \tag{8.4}$$

To see that this is small for x' small, we split the integral into two pieces. Since ψ is very small near infinity, given $\epsilon > 0$ we can choose R so large that

$$\|v\|_\infty \cdot \int_{|y]|\ge R} |\psi(y)|\ dy \le \epsilon/4\,. \tag{8.5}$$

The set $|y| \le R$ is compact and if $|x| \le R'$, $|x'| \le 1$ then $|x+x'-y| \le R+R'+1$. A continuous function is *uniformly continuous* on any compact set, so we can chose $\delta > 0$ such that

$$\sup_{\substack{|x'|<\delta \\ |y|\le R}} |v(x+x'-y) - v(x-y)| \cdot \int_{|y|\le R} |\psi(y)| \, dy \ < \epsilon/2 \, . \tag{8.6}$$

Combining (8.5) and (8.6) we conclude that $v*\psi$ is continuous. Finally, we conclude that

$$v \in \mathcal{C}^0_0(\mathbb{R}^n) \Rightarrow v * \psi \in \mathcal{C}^0_0(\mathbb{R}^n) \, . \tag{8.7}$$

For this we need to show that $v * \psi$ is small at infinity, which follows from the fact that v is small at infinity. Namely given $\epsilon > 0$ there exists $R > 0$ such that $|v(y)| \le \epsilon$ if $|y| \ge R$. Divide the integral defining the convolution into two

$$\begin{aligned} |v * \psi(x)| \le \int_{|y|>R} u(y)\psi(x-y)dy + \int_{y<R} |u(y)\psi(x-y)|dy \\ \le \epsilon/2\|\psi\|_\infty + \|u\|_\infty \sup_{B(x,R)} |\psi|. \end{aligned}$$

Since $\psi \in \mathcal{S}(\mathbb{R}^n)$ the last constant tends to 0 as $|x| \to \infty$.

We can do much better than this! Assuming $|x'| \le 1$ we can use Taylor's formula with remainder to write

$$\psi(z+x') - \psi(z) = \int_0^{t'} \frac{d}{dt}\psi(z+tx')\, dt = \sum_{j=1}^n x_j \cdot \tilde{\psi}_j(z,x') \, . \tag{8.8}$$

As Problem 23 I ask you to check carefully that

$$\psi_j(z;x') \in \mathcal{S}(\mathbb{R}^n) \text{ depends continuously on } x' \text{ in } \ |x'| \le 1 \, . \tag{8.9}$$

Going back to (8.3))we can use the translation and reflection-invariance of Lebesgue measure to rewrite the integral (by changing variable) as

$$v * \psi(x) = \int_{\mathbb{R}^n} v(y)\psi(x-y)\, dy \, . \tag{8.10}$$

This reverses the role of v and ψ and shows that if *both* v and ψ are in $\mathcal{S}(\mathbb{R}^n)$ then $v * \psi = \psi * v$.

Using this formula on (8.4) we find

(8.11)

$$\begin{aligned} v * \psi(x+x') - v * \psi(x) = \int v(y)(\psi(x+x'-y) - \psi(x-y))\, dy \\ = \sum_{j=1}^n x_j \int_{\mathbb{R}^n} v(y)\tilde{\psi}_j(x-y,x')\, dy = \sum_{j=1}^n x_j(v * \psi_j(\cdot;x')(x) \, . \end{aligned}$$

From (8.9) and what we have already shown, $v * \psi(\cdot; x')$ is continuous in both variables, and is in $\mathcal{C}^0_0(\mathbb{R}^n)$ in the first. Thus

$$v \in \mathcal{C}^0_0(\mathbb{R}^n)\,,\ \psi \in \mathcal{S}(\mathbb{R}^n) \Rightarrow v * \psi \in \mathcal{C}^1_0(\mathbb{R}^n)\,. \tag{8.12}$$

In fact we also see that

$$\frac{\partial}{\partial x_j} v * \psi = v * \frac{\partial \psi}{\partial x_j}\,. \tag{8.13}$$

Thus $v * \psi$ inherits its regularity from ψ.

Proposition 8.1. *If $v \in \mathcal{C}^0_0(\mathbb{R}^n)$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$ then*

$$v * \psi \in \mathcal{C}^\infty_0(\mathbb{R}^n) = \bigcap_{k \geq 0} \mathcal{C}^k_0(\mathbb{R}^n)\,. \tag{8.14}$$

Proof. This follows from (8.12), (8.13) and induction. □

Now, let us make a more special choice of ψ. We have shown the existence of

$$\varphi \in \mathcal{C}^\infty_c(\mathbb{R}^n)\,,\ \varphi \geq 0\,,\ \operatorname{supp}(\varphi) \subset \{|x| \leq 1\}\,. \tag{8.15}$$

We can also assume $\int_{\mathbb{R}^n} \varphi\, dx = 1$, by multiplying by a positive constant. Now consider

$$\varphi_t(x) = t^{-n} \varphi\left(\frac{x}{t}\right)\ \ 1 \geq t > 0\,. \tag{8.16}$$

This has all the same properties, except that

$$\operatorname{supp} \varphi_t \subset \{|x| \leq t\}\ ,\ \int \varphi_t\, dx = 1\,. \tag{8.17}$$

Proposition 8.2. *If $v \in \mathcal{C}^0_0(\mathbb{R}^n)$ then as $t \to 0$, $v_t = v * \varphi_t \to v$ in $\mathcal{C}^0_0(\mathbb{R}^n)$.*

Proof. using (8.17) we can write the difference as

$$|v_t(x) - v(x)| = |\int_{\mathbb{R}^n} (v(x-y) - v(x))\varphi_t(y)\, dy| \leq \sup_{|y| \leq t} |v(x-y) - v(x)| \to 0. \tag{8.18}$$

Here we have used the fact that $\varphi_t \geq 0$ has support in $|y| \leq t$ and has integral 1. Thus $v_t \to v$ uniformly on any set on which v is uniformly continuous, namel $\mathbb{R}^n$! □

Corollary 8.3. *$\mathcal{C}^k_0(\mathbb{R}^n)$ is dense in $\mathcal{C}^p_0(\mathbb{R}^n)$ for any $k \geq p$.*

Proposition 8.4. *$\mathcal{S}(\mathbb{R}^n)$ is dense in $\mathcal{C}^k_0(\mathbb{R}^n)$ for any $k \geq 0$.*

Proof. Take $k = 0$ first. The subspace $\mathcal{C}^0_c(\mathbb{R}^n)$ is dense in $\mathcal{C}^0_0(\mathbb{R}^n)$, by cutting off outside a large ball. If $v \in \mathcal{C}^0_c(\mathbb{R}^n)$ has support in $\{|x| \le R\}$ then

$$v * \varphi_t \in \mathcal{C}^\infty_c(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$$

has support in $\{|x| \le R + 1\}$. Since $v * \varphi_t \to v$ the result follows for $k = 0$.

For $k \ge 1$ the same argument works, since $D^\alpha(v * \varphi_t) = (D^\alpha V) * \varphi_t$. □

Corollary 8.5. *The map from finite Radon measures*

$$M_{fin}(\mathbb{R}^n) \ni \mu \longmapsto u_\mu \in \mathcal{S}'(\mathbb{R}^n) \tag{8.19}$$

is injective.

Now, we want the same result for $L^2(\mathbb{R}^n)$ (and maybe for $L^p(\mathbb{R}^n)$, $1 \le p < \infty$). I leave the measure-theoretic part of the argument to you.

Proposition 8.6. *Elements of $L^2(\mathbb{R}^n)$ are "continuous in the mean" i.e.,*

$$\lim_{|t|\to 0} \int_{\mathbb{R}^n} |u(x+t) - u(x)|^2 \, dx = 0\,. \tag{8.20}$$

This is Problem 24.

Using this we conclude that

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n) \text{ is dense} \tag{8.21}$$

as before. First observe that the space of L^2 functions of compact support is dense in $L^2(\mathbb{R}^n)$, since

$$\lim_{R\to\infty} \int_{|x|\ge R} |u(x)|^2 \, dx = 0 \;\forall\; u \in L^2(\mathbb{R}^n)\,.$$

Then look back at the discussion of $v * \varphi$, now v is replaced by $u \in L^2_c(\mathbb{R}^n)$. The compactness of the support means that $u \in L^1(\mathbb{R}^n)$ so in

$$u * \varphi(x) = \int_{\mathbb{R}^n} u(x-y)\varphi(y)dy \tag{8.22}$$

the integral is absolutely convergent. Moreover

$$\begin{aligned} &|u * \varphi(x + x') - u * \varphi(x)| \\ &\qquad = \left| \int u(y)(\varphi(x + x' - y) - \varphi(x - y))\, dy \right| \\ &\qquad\qquad \le C\|u\| \sup_{|y|\le R} |\varphi(x + x' - y) - \varphi(x - y)| \to 0 \end{aligned}$$

when $\{|x| \le R\}$ large enough. Thus $u * \varphi$ is continuous and the same argument as before shows that

$$u * \varphi_t \in \mathcal{S}(\mathbb{R}^n) .$$

Now to see that $u * \varphi_t \to u$, assuming u has compact support (or not) we estimate the integral

$$\begin{aligned} |u * \varphi_t(x) - u(x)| &= \left| \int (u(x-y) - u(x))\varphi_t(y)\, dy \right| \\ &\le \int |u(x-y) - u(x)|\, \varphi_t(y)\, dy . \end{aligned}$$

Using the same argument twice

$$\begin{aligned} &\int |u * \varphi_t(x) - u(x)|^2 \, dx \\ &\le \iiint |u(x-y) - u(x)|\, \varphi_t(y)\, |u(x-y') - u(x)|\, \varphi_t(y')\, dx\, dy\, dy' \\ &\qquad \le \left(\int |u(x-y) - u(x)|^2 \, \varphi_t(y)\varphi_t(y') dx\, dy\, dy' \right) \\ &\qquad\qquad \le \sup_{|y| \le t} \int |u(x-y) - u(x)|^2 \, dx . \end{aligned}$$

Note that at the second step here I have used Schwarz's inequality with the integrand written as the product

$$|u(x-y) - u(x)|\, \varphi_t^{1/2}(y)\varphi_t^{1/2}(y') \cdot |u(x-y') - u(x)|\, \varphi_t^{1/2}(y)\varphi_t^{1/2}(y') .$$

Thus we now know that

$$L^2(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n) \text{ is injective.}$$

This means that all our usual spaces of functions 'sit inside' $\mathcal{S}'(\mathbb{R}^n)$.

Finally we can use convolution with φ_t to show the existence of *smooth* partitions of unity. If $K \Subset U \subset \mathbb{R}^n$ is a compact set in an open set then we have shown the existence of $\xi \in \mathcal{C}_c^0(\mathbb{R}^n)$, with $\xi = 1$ in some neighborhood of K and $\xi = 1$ in some neighborhood of K and $\operatorname{supp}(\xi) \Subset U$.

Then consider $\xi * \varphi_t$ for t small. In fact

$$\operatorname{supp}(\xi * \varphi_t) \subset \{p \in \mathbb{R}^n \,;\, \operatorname{dist}(p, \operatorname{supp} \xi) \le 2t\}$$

and similarly, $0 \le \xi * \varphi_t \le 1$ and

$$\xi * \varphi_t = 1 \text{ at } p \text{ if } \xi = 1 \text{ on } B(p, 2t) .$$

Using this we get:

Proposition 8.7. *If $U_a \subset \mathbb{R}^n$ are open for $a \in A$ and $K \Subset \bigcup_{a\in A} U_a$ then there exist finitely many $\varphi_i \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, with $0 \le \varphi_i \le 1$, $\operatorname{supp}(\varphi_i) \subset U_{a_i}$ such that $\sum_i \varphi_i = 1$ in a neighbourhood of K.*

Proof. By the compactness of K we may choose a finite open subcover. Using Lemma 1.8 we may choose a continuous partition, ϕ_i', of unity subordinate to this cover. Using the convolution argument above we can replace ϕ_i' by $\phi_i' * \varphi_t$ for $t > 0$. If t is sufficiently small then this is again a partition of unity subordinate to the cover, but now smooth. □

Next we can make a simple 'cut off argument' to show

Lemma 8.8. *The space $\mathcal{C}_c^\infty(\mathbb{R}^n)$ of $\mathcal{C}^\infty$ functions of compact support is dense in $\mathcal{S}(\mathbb{R}^n)$.*

Proof. Choose $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ with $\varphi(x) = 1$ in $|x| \le 1$. Then given $\psi \in \mathcal{S}(\mathbb{R}^n)$ consider the sequence

$$\psi_n(x) = \varphi(x/n)\psi(x)\,.$$

Clearly $\psi_n = \psi$ on $|x| \le n$, so if it converges in $\mathcal{S}(\mathbb{R}^n)$ it must converge to ψ. Suppose $m \ge n$ then by Leibniz's formula[13]

$$D_x^\alpha(\psi_n(x) - \psi_m(x)) = \sum_{\beta\le\alpha} \binom{\alpha}{\beta} D_x^\beta \left(\varphi(\frac{x}{n}) - \varphi(\frac{x}{m})\right) \cdot D_x^{\alpha-\beta}\psi(x)\,.$$

All derivatives of $\varphi(x/n)$ are bounded, independent of n and $\psi_n = \psi_m$ in $|x| \le n$ so for any p

$$|D_x^\alpha(\psi_n(x) - \psi_m(x))| \le \begin{cases} 0 & |x| \le n \\ C_{\alpha,p}\langle x\rangle^{-2p} & |x| \ge n \end{cases}.$$

Hence ψ_n is Cauchy in $\mathcal{S}(\mathbb{R}^n)$. □

Thus every element of $\mathcal{S}'(\mathbb{R}^n)$ is determined by its restriction to $\mathcal{C}_c^\infty(\mathbb{R}^n)$. The support of a tempered distribution was defined above to be

$$\operatorname{supp}(u) = \{x \in \mathbb{R}^n;\ \exists\, \varphi \in \mathcal{S}(\mathbb{R}^n)\,,\ \varphi(x) \ne 0\,,\ \varphi u = 0\}^{\complement}\,. \tag{8.23}$$

Using the preceding lemma and the construction of smooth partitions of unity we find

Proposition 8.9. *f $u \in \mathcal{S}'(\mathbb{R}^n)$ and* $\operatorname{supp}(u) = \emptyset$ *then $u = 0$.*

[13]Problem 25.

Proof. From (8.23), if $\psi \in \mathcal{S}(\mathbb{R}^n)$, $\operatorname{supp}(\psi u) \subset \operatorname{supp}(u)$. If $x \ni \operatorname{supp}(u)$ then, by definition, $\varphi u = 0$ for some $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with $\varphi(x) \neq 0$. Thus $\varphi \neq 0$ on $B(x, \epsilon)$ for $\epsilon > 0$ sufficiently small. If $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ has support in $B(x, \epsilon)$ then $\psi u = \tilde{\psi}\varphi u = 0$, where $\tilde{\psi} \in \mathcal{C}_c^\infty(\mathbb{R}^n)$:

$$\tilde{\psi} = \begin{cases} \psi/\varphi & \text{in } B(x, \epsilon) \\ 0 & \text{elsewhere}. \end{cases}$$

Thus, given $K \Subset \mathbb{R}^n$ we can find $\varphi_j \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, supported in such balls, so that $\sum_j \varphi_j \equiv 1$ on K but $\varphi_j u = 0$. For given $\mu \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ apply this to $\operatorname{supp}(\mu)$. Then

$$\mu = \sum_j \varphi_j \mu \Rightarrow u(\mu) = \sum_j (\phi_j u)(\mu) = 0\,.$$

Thus $u = 0$ on $\mathcal{C}_c^\infty(\mathbb{R}^n)$, so $u = 0$. □

The linear space of distributions of compact support will be denoted $\mathcal{C}_c^{-\infty}(\mathbb{R}^n)$; it is often written $\mathcal{E}'(\mathbb{R}^n)$.

Now let us give a characterization of the 'delta function'

$$\delta(\varphi) = \varphi(0) \ \forall \ \varphi \in \mathcal{S}(\mathbb{R}^n)\,,$$

or at least the one-dimensional subspace of $\mathcal{S}'(\mathbb{R}^n)$ it spans. This is based on the simple observation that $(x_j\varphi)(0) = 0$ if $\varphi \in \mathcal{S}(\mathbb{R}^n)$!

Proposition 8.10. *If $u \in \mathcal{S}'(\mathbb{R}^n)$ satisfies $x_j u = 0$, $j = 1, \cdots, n$ then $u = c\delta$.*

Proof. The main work is in characterizing the null space of δ as a linear functional, namely in showing that

$$\mathcal{H} = \{\varphi \in \mathcal{S}(\mathbb{R}^n);\ \varphi(0) = 0\} \tag{8.24}$$

can also be written as

$$\mathcal{H} = \left\{\varphi \in \mathcal{S}(\mathbb{R}^n);\ \varphi = \sum_{j=1}^n x_j \psi_j\,,\ \varphi_j \in \mathcal{S}(\mathbb{R}^n)\right\}. \tag{8.25}$$

Clearly the right side of (8.25) is contained in the left. To see the converse, suppose first that

$$\varphi \in \mathcal{S}(\mathbb{R}^n)\,,\ \varphi = 0 \text{ in } |x| < 1\,. \tag{8.26}$$

Then define

$$\psi = \begin{cases} 0 & |x| < 1 \\ \varphi/\left|x\right|^2 & |x| \geq 1\,. \end{cases}$$

All the derivatives of $1/|x|^2$ are bounded in $|x| \geq 1$, so from Leibniz's formula it follows that $\psi \in \mathcal{S}(\mathbb{R}^n)$. Since

$$\varphi = \sum_j x_j(x_j\psi)$$

this shows that φ of the form (8.26) is in the right side of (8.25). In general suppose $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Then

$$\begin{aligned} \varphi(x) - \varphi(0) &= \int_0^t \frac{d}{dt}\varphi(tx)\,dt \\ &= \sum_{j=1}^n x_j \int_0^t \frac{\partial\varphi}{\partial x_j}(tx)\,dt\,. \end{aligned} \tag{8.27}$$

Certainly these integrals are $\mathcal{C}^\infty$, but they may not decay rapidly at infinity. However, choose $\mu \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ with $\mu = 1$ in $|x| \leq 1$. Then (8.27) becomes, if $\varphi(0) = 0$,

$$\begin{aligned} \varphi &= \mu\varphi + (1-\mu)\varphi \\ &= \sum_{j=1}^n x_j\psi_j + (1-\mu)\varphi\,,\ \psi_j = \mu\int_0^t \frac{\partial\varphi}{\partial x_j}(tx)\,dt \in \mathcal{S}(\mathbb{R}^n)\,. \end{aligned}$$

Since $(1-\mu)\varphi$ is of the form (8.26), this proves (8.25).

Our assumption on u is that $x_j u = 0$, thus

$$u(\varphi) = 0\ \forall\ \varphi \in \mathcal{H}$$

by (8.25). Choosing μ as above, a general $\varphi \in \mathcal{S}(\mathbb{R}^n)$ can be written

$$\varphi = \varphi(0)\cdot\mu + \varphi'\,,\ \varphi' \in \mathcal{H}\,.$$

Then

$$u(\varphi) = \varphi(0)u(\mu) \Rightarrow u = c\delta\,,\ c = u(\mu)\,.$$

□

This result is quite powerful, as we shall soon see. The Fourier transform of an element $\varphi \in \mathcal{S}(\mathbb{R}^n)$ is[14]

$$\hat\varphi(\xi) = \int e^{-ix\cdot\xi}\varphi(x)\,dx\,,\ \xi \in \mathbb{R}^n\,. \tag{8.28}$$

[14]Normalizations vary, but it doesn't matter much.

The integral certainly converges, since $|\varphi| \le C\langle x\rangle^{-n-1}$. In fact it follows easily that $\hat\varphi$ is continuous, since

$$|\hat\varphi(\xi) - \hat\varphi(\xi')| \in \int \left| e^{ix-\xi} - e^{-x\cdot\xi'} \right| |\varphi|\, dx \\ \to 0 \text{ as } \xi' \to \xi\,.$$

In fact

Proposition 8.11. *Fourier transformation,* (8.28)*, defines a continuous linear map*

$$\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)\,,\ \mathcal{F}\varphi = \hat\varphi\,. \tag{8.29}$$

Proof. Differentiating under the integral[15] sign shows that

$$\partial_{\xi_j}\hat\varphi(\xi) = -i\int e^{-ix\cdot\xi}x_j\varphi(x)\,dx\,.$$

Since the integral on the right *is* absolutely convergent that shows that (remember the i's)

$$D_{\xi_j}\hat\varphi = -\widehat{x_j\varphi}\,,\ \forall\ \varphi \in \mathcal{S}(\mathbb{R}^n)\,. \tag{8.30}$$

Similarly, if we multiply by ξ_j and observe that $\xi_j e^{-ix\cdot\xi} = i\frac{\partial}{\partial x_j}e^{-ix\cdot\xi}$ then integration by parts shows

$$\begin{aligned} \xi_j\hat\varphi &= i\int (\frac{\partial}{\partial x_j}e^{-ix\cdot\xi})\varphi(x)\,dx \\ &= -i\int e^{-ix\cdot\xi}\frac{\partial\varphi}{\partial x_j}\,dx \\ \widehat{D_j\varphi} &= \xi_j\hat\varphi\,,\ \forall\ \varphi \in \mathcal{S}(\mathbb{R}^n)\,. \end{aligned} \tag{8.31}$$

Since $x_j\varphi$, $D_j\varphi \in \mathcal{S}(\mathbb{R}^n)$ these results can be iterated, showing that

$$\xi^\alpha D_\xi^\beta\hat\varphi = \mathcal{F}\left((-1)^{|\beta|}D^\alpha{}_x x^\beta\varphi\right)\,. \tag{8.32}$$

Thus $\left|\xi^\alpha D_\xi^\beta\hat\varphi\right| \le C_{\alpha\beta}\sup\left|\langle x\rangle^{+n+1}D^\alpha{}_x x^\beta\varphi\right| \le C\|\langle x\rangle^{n+1+|\beta|}\varphi\|_{\mathcal{C}^{|\alpha|}}$, which shows that $\mathcal{F}$ is continuous as a map (8.32).

□

Suppose $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Since $\hat\varphi \in \mathcal{S}(\mathbb{R}^n)$ we can consider the distribution $u \in \mathcal{S}'(\mathbb{R}^n)$

$$u(\varphi) = \int_{\mathbb{R}^n}\hat\varphi(\xi)\,d\xi\,. \tag{8.33}$$

[15] See [5]

The continuity of u follows from the fact that integration is continuous and (8.29). Now observe that

$$\begin{aligned} u(x_j\varphi) &= \int_{\mathbb{R}^n} \widehat{x_j\varphi}(\xi)\,d\xi \\ &= -\int_{\mathbb{R}^n} D_{\xi_j}\hat{\varphi}\,d\xi = 0 \end{aligned}$$

where we use (8.30). Applying Proposition 8.10 we conclude that $u = c\delta$ for some (universal) constant c. By definition this means

$$\int_{\mathbb{R}^n} \hat{\varphi}(\xi)\,d\xi = c\varphi(0)\,. \tag{8.34}$$

So what is the constant? To find it we need to work out an example. The simplest one is

$$\varphi = \exp(-|x|^2/2)\,.$$

Lemma 8.12. *The Fourier transform of the Gaussian* $\exp(-|x|^2/2)$ *is the Gaussian* $(2\pi)^{n/2}\exp(-|\xi|^2/2)$.

Proof. There are two obvious methods — one uses complex analysis (Cauchy's theorem) the other, which I shall follow, uses the uniqueness of solutions to ordinary differential equations.

First observe that $\exp(-|x|^2/2) = \prod_j \exp(-x_j^2/2)$. Thus[16]

$$\hat{\varphi}(\xi) = \prod_{j=1}^{n} \hat{\psi}(\xi_j)\,,\ \psi(x) = e^{-x^2/2}\,,$$

being a function of one variable. Now ψ satisfies the differential equation

$$(\partial_x + x)\,\psi = 0\,,$$

and is the *only* solution of this equation up to a constant multiple. By (8.30) and (8.31) its Fourier transform satisfies

$$\widehat{\partial_x\psi} + \widehat{x\psi} = i\xi\hat{\psi} + i\frac{d}{d\xi}\hat{\varphi} = 0\,.$$

This is the same equation, but in the ξ variable. Thus $\hat{\psi} = ce^{-|\xi|^2/2}$. Again we need to find the constant. However,

$$\hat{\psi}(0) = c = \int e^{-x^2/2}\,dx = (2\pi)^{1/2}$$

[16]Really by Fubini's theorem, but here one can use Riemann integrals.

by the standard use of polar coordinates:

$$c^2 = \int_{\mathbb{R}^n} e^{-(x^2+y^2)/2}\, dx\, dy = \int_0^\infty \int_0^{2\pi} e^{-r^2/2} r\, dr\, d\theta = 2\pi\,.$$

This proves the lemma.

□

Thus we have shown that for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \hat{\varphi}(\xi)\, d\xi = (2\pi)^n \varphi(0)\,. \tag{8.35}$$

Since this is true for $\varphi = \exp(-|x|^2/2)$. The identity allows us to *invert* the Fourier transform.