7. Tempered distributions

A good first reference for distributions is [2], [4] gives a more exhaustive treatment.

The complete metric topology on $\mathcal{S}(\mathbb{R}^n)$ is described above. Next I want to try to convice you that elements of its dual space $\mathcal{S}'(\mathbb{R}^n)$, have enough of the properties of functions that we can work with them as 'generalized functions'.

First let me develop some notation. A differentiable function $\varphi : \mathbb{R}^n \to \mathbb{C}$ has partial derivatives which we have denoted $\partial\varphi/\partial x_j : \mathbb{R}^n \to \mathbb{C}$. For reasons that will become clear later, we put a $\sqrt{-1}$ into the definition and write

$$D_j\varphi = \frac{1}{i}\frac{\partial\varphi}{\partial x_j}\,. \tag{7.1}$$

We say φ is once continuously differentiable if each of these $D_j\varphi$ is continuous. Then we defined k times continuous differentiability inductively by saying that φ and the $D_j\varphi$ are $(k-1)$-times continuously differentiable. For $k=2$ this means that

$$D_jD_k\varphi \text{ are continuous for } j, k = 1, \cdots, n\,.$$

Now, recall that, if continuous, these second derivatives are symmetric:

$$D_jD_k\varphi = D_kD_j\varphi\,. \tag{7.2}$$

This means we can use a compact notation for higher derivatives. Put $\mathbb{N}_0 = \{0, 1, \ldots\}$; we call an element $\alpha \in \mathbb{N}_0^n$ a 'multi-index' and if φ is at least k times continuously differentiable, we set[12]

$$D^\alpha\varphi = \frac{1}{i^{|\alpha|}}\frac{\partial^{\alpha_1}}{\partial x_1}\cdots\frac{\partial^{\alpha_n}}{\partial x_n}\varphi \text{ whenever } |\alpha| = \alpha_1+\alpha_2+\cdots+\alpha_n \le k. \tag{7.3}$$

Now we have *defined* the spaces.

$$\mathcal{C}_0^k(\mathbb{R}^n) = \left\{\varphi : \mathbb{R}^n \to \mathbb{C}\,;\, D^\alpha\varphi \in \mathcal{C}_0^0(\mathbb{R}^n)\ \forall\ |\alpha| \le k\right\}\,. \tag{7.4}$$

Notice the convention is that $D^\alpha\varphi$ is asserted to exist if it is required to be continuous! Using $\langle x\rangle = (1+|x|^2)$ we defined

$$\langle x\rangle^{-k}\mathcal{C}_0^k(\mathbb{R}^n) = \left\{\varphi : \mathbb{R}^n \to \mathbb{C}\,;\, \langle x\rangle^k\varphi \in \mathcal{C}_0^k(\mathbb{R}^n)\right\}\,, \tag{7.5}$$

and then our space of test functions is

$$\mathcal{S}(\mathbb{R}^n) = \bigcap_k \langle x\rangle^{-k}\mathcal{C}_0^k(\mathbb{R}^n)\,.$$

[12]Periodically there is the possibility of confusion between the two meanings of $|\alpha|$ but it seldom arises.

Thus,

$$\varphi \in \mathcal{S}(\mathbb{R}^n) \Leftrightarrow D^\alpha(\langle x\rangle^k \varphi) \in \mathcal{C}^0_0(\mathbb{R}^n)\ \forall\ |\alpha| \le k \text{ and all } k\,. \tag{7.6}$$

Lemma 7.1. *The condition $\varphi \in \mathcal{S}(\mathbb{R}^n)$ can be written*

$$\langle x\rangle^k D^\alpha \varphi \in \mathcal{C}^0_0(\mathbb{R}^n)\ \forall\ |\alpha| \le k\,,\ \forall\ k\,.$$

Proof. We first check that

$$\begin{aligned} &\varphi \in \mathcal{C}^0_0(\mathbb{R}^n)\,,\ D_j(\langle x\rangle\varphi) \in \mathcal{C}^0_0(\mathbb{R}^n)\,,\ j = 1,\cdots,n \\ \Leftrightarrow\ &\varphi \in \mathcal{C}^0_0(\mathbb{R}^n)\,,\ \langle x\rangle D_j\varphi \in \mathcal{C}^0_0(\mathbb{R}^n)\,,\ j = 1,\cdots,n\,. \end{aligned}$$

Since

$$D_j\langle x\rangle\varphi = \langle x\rangle D_j\varphi + (D_j\langle x\rangle)\varphi$$

and $D_j\langle x\rangle = \frac{1}{i}x_j\langle x\rangle^{-1}$ is a bounded continuous function, this is clear. Then consider the same thing for a larger k:

$$\begin{aligned} &D^\alpha\langle x\rangle^p\varphi \in \mathcal{C}^0_0(\mathbb{R}^n)\ \forall\ |\alpha| = p\,,\ 0 \le p \le k \\ \Leftrightarrow\ &\langle x\rangle^p D^\alpha\varphi \in \mathcal{C}^0_0(\mathbb{R}^n)\ \forall\ |\alpha| = p\,,\ 0 \le p \le k\,. \end{aligned} \tag{7.7}$$

□

I leave you to check this as Problem 7.1.

Corollary 7.2. *For any $k \in \mathbb{N}$ the norms*

$$\|\langle x\rangle^k\varphi\|_{\mathcal{C}^k} \text{ and } \sum_{\substack{|\alpha|\le k,\\ |\beta|\le k}} \|x^\alpha D^\beta_x\varphi\|_\infty$$

are equivalent.

Proof. Any reasonable proof of (7.2) shows that the norms

$$\|\langle x\rangle^k\varphi\|_{\mathcal{C}^k} \text{ and } \sum_{|\beta|\le k} \|\langle x\rangle^k D^\beta\varphi\|_\infty$$

are equivalent. Since there are positive constants such that

$$C_1\left(1 + \sum_{|\alpha|\le k}|x^\alpha|\right) \le \langle x\rangle^k \le C_2\left(1 + \sum_{|\alpha|\le k}|x^\alpha|\right)$$

the equivalent of the norms follows.

□

Proposition 7.3. *A linear functional $u : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ is continuous if and only if there exist C, k such that*

$$|u(\varphi)| \le C \sum_{\substack{|\alpha|\le k,\\ |\beta|\le k}} \sup_{\mathbb{R}^n} \left|x^\alpha D^\beta_x\varphi\right|.$$

Proof. This is just the equivalence of the norms, since we showed that $u \in \mathcal{S}'(\mathbb{R}^n)$ if and only if

$$|u(\varphi)| \le C \|\langle x \rangle^k \varphi\|_{\mathcal{C}^k}$$

for some k.

□

Lemma 7.4. *A linear* map

$$T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$$

is continuous if and only if for each k *there exist* C *and* j *such that if* $|\alpha| \le k$ *and* $|\beta| \le k$

$$\sup \left| x^\alpha D^\beta T\varphi \right| \le C \sum_{|\alpha'| \le j,\ |\beta'| \le j} \sup_{\mathbb{R}^n} \left| x^{\alpha'} D^{\beta'} \varphi \right| \ \forall\ \varphi \in \mathcal{S}(\mathbb{R}^n). \tag{7.8}$$

Proof. This is Problem 7.2. □

All this messing about with norms shows that

$$x_j : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n) \text{ and } D_j : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$$

are continuous.

So now we have some idea of what $u \in \mathcal{S}'(\mathbb{R}^n)$ means. Let's notice that $u \in \mathcal{S}'(\mathbb{R}^n)$ implies

$$x_j u \in \mathcal{S}'(\mathbb{R}^n)\ \forall\ j = 1, \cdots, n \tag{7.9}$$

$$D_j u \in \mathcal{S}'(\mathbb{R}^n)\ \forall\ j = 1, \cdots, n \tag{7.10}$$

$$\varphi u \in \mathcal{S}'(\mathbb{R}^n)\ \forall\ \varphi \in \mathcal{S}(\mathbb{R}^n) \tag{7.11}$$

where we have to *define* these things in a reasonable way. Remember that $u \in \mathcal{S}'(\mathbb{R}^n)$ is "supposed" to be like an integral against a "generalized function"

$$u(\psi) = \int_{\mathbb{R}^n} u(x)\psi(x)\, dx\ \forall\ \psi \in \mathcal{S}(\mathbb{R}^n). \tag{7.12}$$

Since it would be true if u *were* a function we *define*

$$x_j u(\psi) = u(x_j \psi)\ \forall\ \psi \in \mathcal{S}(\mathbb{R}^n). \tag{7.13}$$

Then we check that $x_j u \in \mathcal{S}'(\mathbb{R}^n)$:

$$\begin{aligned} |x_j u(\psi)| &= |u(x_j\psi)| \\ &\le C \sum_{|\alpha| \le k,\ |\beta| \le k} \sup_{\mathbb{R}^n} \left| x^\alpha D^\beta (x_j \psi) \right| \\ &\le C' \sum_{|\alpha| \le k+1,\ |\beta| \le k} \sup_{\mathbb{R}^n} \left| x^\alpha D^\beta \psi \right| . \end{aligned}$$

Similarly we can define the partial *derivatives* by using the standard integration by parts formula

$$\int_{\mathbb{R}^n}(D_j u)(x)\varphi(x)\,dx = -\int_{\mathbb{R}^n} u(x)(D_j\varphi(x))\,dx \tag{7.14}$$

if $u \in \mathcal{C}_0^1(\mathbb{R}^n)$. Thus if $u \in \mathcal{S}'(\mathbb{R}^n)$ again we *define*

$$D_j u(\psi) = -u(D_j\psi)\ \forall\ \psi \in \mathcal{S}(\mathbb{R}^n).$$

Then it is clear that $D_j u \in \mathcal{S}'(\mathbb{R}^n)$.

Iterating these definition we find that D^α, for any multi-index α, defines a linear map

$$D^\alpha : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)\,. \tag{7.15}$$

In general a linear differential operator with constant coefficients is a sum of such "monomials". For example Laplace's operator is

$$\Delta = -\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \cdots - \frac{\partial^2}{\partial x_n^2} = D_1^2 + D_2^2 + \cdots + D_n^2\,.$$

We will be interested in trying to solve differential equations such as

$$\Delta u = f \in \mathcal{S}'(\mathbb{R}^n)\,.$$

We can also multiply $u \in \mathcal{S}'(\mathbb{R}^n)$ by $\varphi \in \mathcal{S}(\mathbb{R}^n)$, simply defining

$$\varphi u(\psi) = u(\varphi\psi)\ \forall\ \psi \in \mathcal{S}(\mathbb{R}^n). \tag{7.16}$$

For this to make sense it suffices to check that

$$\sum_{\substack{|\alpha|\le k,\\ |\beta|\le k}} \sup_{\mathbb{R}^n}\left|x^\alpha D^\beta(\varphi\psi)\right| \le C \sum_{\substack{|\alpha|\le k,\\ |\beta|\le k}} \sup_{\mathbb{R}^n}\left|x^\alpha D^\beta\psi\right|\,. \tag{7.17}$$

This follows easily from Leibniz' formula.

Now, to start thinking of $u \in \mathcal{S}'(\mathbb{R}^n)$ as a generalized function we first define its *support*. Recall that

$$\operatorname{supp}(\psi) = \operatorname{clos}\{x \in \mathbb{R}^n; \psi(x) \neq 0\}\,. \tag{7.18}$$

We can write this in another 'weak' way which is easier to generalize. Namely

$$p \notin \operatorname{supp}(u) \Leftrightarrow\ \exists\varphi \in \mathcal{S}(\mathbb{R}^n)\,,\ \varphi(p) \neq 0\,,\ \varphi u = 0\,. \tag{7.19}$$

In fact this definition makes sense for *any* $u \in \mathcal{S}'(\mathbb{R}^n)$.

Lemma 7.5. *The set* $\operatorname{supp}(u)$ *defined by* (7.19) *is a closed subset of* $\mathbb{R}^n$ *and reduces to* (7.18) *if* $u \in \mathcal{S}(\mathbb{R}^n)$.

Proof. The set defined by (7.19) is closed, since

$$\operatorname{supp}(u)^{\complement} = \{p \in \mathbb{R}^n;\ \exists\ \varphi \in \mathcal{S}(\mathbb{R}^n),\ \varphi(p) \neq 0,\ \varphi u = 0\} \tag{7.20}$$

is clearly open — the same φ works for nearby points. If $\psi \in \mathcal{S}(\mathbb{R}^n)$ we define $u_\psi \in \mathcal{S}'(\mathbb{R}^n)$, which we will again identify with ψ, by

$$u_\psi(\varphi) = \int \varphi(x)\psi(x)\,dx\,. \tag{7.21}$$

Obviously $u_\psi = 0 \Longrightarrow \psi = 0$, simply set $\varphi = \overline{\psi}$ in (7.21). Thus the map

$$\mathcal{S}(\mathbb{R}^n) \ni \psi \longmapsto u_\psi \in \mathcal{S}'(\mathbb{R}^n) \tag{7.22}$$

is injective. We want to show that

$$\operatorname{supp}(u_\psi) = \operatorname{supp}(\psi) \tag{7.23}$$

on the left given by (7.19) and on the right by (7.18). We show first that

$$\operatorname{supp}(u_\psi) \subset \operatorname{supp}(\psi).$$

Thus, we need to see that $p \notin \operatorname{supp}(\psi) \Rightarrow p \notin \operatorname{supp}(u_\psi)$. The first condition is that $\psi(x) = 0$ in a neighbourhood, U of p, hence there is a $\mathcal{C}^\infty$ function φ with support in U and $\varphi(p) \neq 0$. Then $\varphi\psi \equiv 0$. Conversely suppose $p \notin \operatorname{supp}(u_\psi)$. Then there exists $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with $\varphi(p) \neq 0$ and $\varphi u_\psi = 0$, i.e., $\varphi u_\psi(\eta) = 0\ \forall\ \eta \in \mathcal{S}(\mathbb{R}^n)$. By the injectivity of $\mathcal{S}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$ this means $\varphi\psi = 0$, so $\psi \equiv 0$ in a neighborhood of p and $p \notin \operatorname{supp}(\psi)$. □

Consider the simplest examples of distribution which are not functions, namely those with support at a given point p. The obvious one is the Dirac delta 'function'

$$\delta_p(\varphi) = \varphi(p)\ \forall\ \varphi \in \mathcal{S}(\mathbb{R}^n)\,. \tag{7.24}$$

We can make many more, because D^α is *local*

$$\operatorname{supp}(D^\alpha u) \subset \operatorname{supp}(u)\ \forall\ u \in \mathcal{S}'(\mathbb{R}^n)\,. \tag{7.25}$$

Indeed, $p \notin \operatorname{supp}(u) \Rightarrow \exists\ \varphi \in \mathcal{S}(\mathbb{R}^n)$, $\varphi u \equiv 0$, $\varphi(p) \neq 0$. Thus each of the distributions $D^\alpha \delta_p$ also has support contained in $\{p\}$. In fact none of them vanish, and they are all linearly independent.