5. Hilbert space

We have shown that $L^p(X,\mu)$ is a Banach space – a complete normed space. I shall next discuss the class of Hilbert spaces, a special class of Banach spaces, of which $L^2(X,\mu)$ is a standard example, in which the norm arises from an inner product, just as it does in Euclidean space.

An inner product on a vector space V over $\mathbb{C}$ (one can do the real case too, not much changes) is a *sesquilinear* form

$$V \times V \to \mathbb{C}$$

written (u,v), if $u,v \in V$. The 'sesqui-' part is just linearity in the first variable

$$(a_1u_1 + a_2u_2\,,\ v) = a_1(u_1,v) + a_2(u_2,v), \tag{5.1}$$

anti-linearly in the second

$$(u, a_1v_1 + a_2v_2) = \overline{a}_1(u,v_1) + \overline{a}_2(u,v_2) \tag{5.2}$$

and the conjugacy condition

$$(u,v) = \overline{(v,u)}\,. \tag{5.3}$$

Notice that (5.2) follows from (5.1) and (5.3). If we assume in addition the positivity condition[8]

$$(u,u) \geq 0\,,\ (u,u) = 0 \Rightarrow u = 0\,, \tag{5.4}$$

then

$$\|u\| = (u,u)^{1/2} \tag{5.5}$$

is a *norm* on V, as we shall see.

Suppose that $u,v \in V$ have $\|u\| = \|v\| = 1$. Then $(u,v) = e^{i\theta}\,|(u,v)|$ for some $\theta \in \mathbb{R}$. By choice of θ, $e^{-i\theta}(u,v) = |(u,v)|$ is real, so expanding out using linearity for $s \in \mathbb{R}$,

$$\begin{aligned} 0 \leq {} & (e^{-i\theta}u - sv\,,\ e^{-i\theta}u - sv) \\ & = \|u\|^2 - 2s\,\mathrm{Re}\ e^{-i\theta}(u,v) + s^2\|v\|^2 = 1 - 2s|(u,v)| + s^2. \end{aligned}$$

The minimum of this occurs when $s = |(u,v)|$ and this is negative unless $|(u,v)| \leq 1$. Using linearity, and checking the trivial cases $u =$ or $v = 0$ shows that

$$|(u,v)| \leq \|u\|\,\|v\|,\ \forall\ u,v \in V\,. \tag{5.6}$$

This is called Schwarz'[9] inequality.

[8]Notice that (u,u) is real by (5.3).

[9]No 't' in this Schwarz.

Using Schwarz' inequality

$$\begin{aligned}\|u+v\|^2 &= \|u\|^2 + (u,v) + (v,u) + \|v\|^2 \\ &\le (\|u\| + \|v\|)^2 \\ &\Longrightarrow \|u+v\| \le \|u\| + \|v\| \ \forall\ u,v \in V\end{aligned}$$

which is the triangle inequality.

Definition 5.1. *A Hilbert space is a vector space V with an inner product satisfying* (5.1) - (5.4) *which is complete as a normed space (i.e., is a Banach space).*

Thus we have already shown $L^2(X,\mu)$ to be a Hilbert space for any positive measure μ. The inner product is

$$(f,g) = \int_X f\overline{g}\, d\mu\,, \tag{5.7}$$

since then (5.3) gives $\|f\|_2$.

Another important identity valid in any inner product spaces is the parallelogram law:

$$\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2\,. \tag{5.8}$$

This can be used to prove the basic 'existence theorem' in Hilbert space theory.

Lemma 5.2. *Let $C \subset H$, in a Hilbert space, be closed and convex (i.e., $su + (1-s)v \in C$ if $u, v \in C$ and $0 < s < 1$). Then C contains a unique element of smallest norm.*

Proof. We can certainly choose a sequence $u_n \in C$ such that

$$\|u_n\| \to \delta = \inf\left\{\|v\|\,;\, v \in C\right\}\,.$$

By the parallelogram law,

$$\begin{aligned}\|u_n - u_m\|^2 &= 2\|u_n\|^2 + 2\|u_m\|^2 - \|u_n + u_m\|^2 \\ &\le 2(\|u_n\|^2 + \|u_m\|^2) - 4\delta^2\end{aligned}$$

where we use the fact that $(u_n + u_m)/2 \in C$ so must have norm at least δ. Thus $\{u_n\}$ is a Cauchy sequence, hence convergent by the assumed completeness of H. Thus $\lim u_n = u \in C$ (since it is assumed closed) and by the triangle inequality

$$|\|u_n\| - \|u\|| \le \|u_n - u\| \to 0$$

So $\|u\| = \delta$. Uniqueness of u follows again from the parallelogram law which shows that if $\|u'\| = \delta$ then

$$\|u - u'\| \le 2\delta^2 - 4\|(u+u')/2\|^2 \le 0\,.$$

□

The fundamental fact about a Hilbert space is that each element $v \in H$ defines a continuous linear functional by

$$H \ni u \longmapsto (u, v) \in \mathbb{C}$$

and conversely *every* continuous linear functional arises this way. This is also called the Riesz representation theorem.

Proposition 5.3. *If $L : H \to \mathbb{C}$ is a continuous linear functional on a Hilbert space then this is a unique element $v \in H$ such that*

$$Lu = (u, v) \ \forall \ u \in H\,, \tag{5.9}$$

Proof. Consider the linear space

$$M = \{u \in H\,;\, Lu = 0\}$$

the null space of L, a continuous linear functional on H. By the assumed continuity, M is closed. We can suppose that L is *not* identically zero (since then $v = 0$ in (5.9)). Thus there exists $w \notin M$. Consider

$$w + M = \{v \in H\,;\, v = w + u\,,\, u \in M\}\ .$$

This is a closed convex subset of H. Applying Lemma 5.2 it has a unique smallest element, $v \in w + M$. Since v minimizes the norm on $w + M$,

$$\|v + su\|^2 = \|v\|^2 + 2\operatorname{Re}(su, v) + \|s\|^2\|u\|^2$$

is stationary at $s = 0$. Thus $\operatorname{Re}(u, v) = 0 \ \forall \ u \in M$, and the same argument with s replaced by is shows that $(v, u) = 0 \ \forall \ u \in M$.

Now $v \in w + M$, so $Lv = Lw \neq 0$. Consider the element $w' = w/Lw \in H$. Since $Lw' = 1$, for any $u \in H$

$$L(u - (Lu)w') = Lu - Lu = 0\,.$$

It follows that $u - (Lu)w' \in M$ so if $w'' = w'/\|w'\|^2$

$$(u, w'') = ((Lu)w', w'') = Lu\frac{(w', w')}{\|w'\|^2} = Lu\,.$$

The uniqueness of v follows from the positivity of the norm. □

Corollary 5.4. *For any positive measure μ, any continuous linear functional*

$$L : L^2(X, \mu) \to \mathbb{C}$$

is of the form

$$Lf = \int_X f\overline{g}\,d\mu\,,\ \ g \in L^2(X, \mu)\,.$$

Notice the apparent power of 'abstract reasoning' here! Although we seem to have constructed g out of nowhere, its existence follows from the *completeness* of $L^2(X, \mu)$, but it is very convenient to express the argument abstractly for a general Hilbert space.