3. Measureability of functions

Suppose that $\mathcal{M}$ is a σ-algebra on a set X[4] and $\mathcal{N}$ is a σ-algebra on another set Y. A map $f : X \to Y$ is said to be *measurable* with respect to these given σ-algebras on X and Y if

$$f^{-1}(E) \in \mathcal{M} \;\forall\; E \in \mathcal{N}\,. \tag{3.1}$$

Notice how similar this is to one of the characterizations of continuity for maps between metric spaces in terms of open sets. Indeed this analogy yields a useful result.

Lemma 3.1. *If $G \subset \mathcal{N}$ generates $\mathcal{N}$, in the sense that*

$$\mathcal{N} = \bigcap\{\mathcal{N}'; \mathcal{N}' \supset G,\ \mathcal{N}' \text{ a } \sigma\text{-algebra}\} \tag{3.2}$$

then $f : X \longrightarrow Y$ is measurable iff $f^{-1}(A) \in \mathcal{M}$ for all $A \in G$.

Proof. The main point to note here is that f^{-1} as a map on power sets, is very well behaved for *any* map. That is if $f : X \to Y$ then $f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X)$ satisfies:

$$\begin{gathered}
f^{-1}(E^C) = (f^{-1}(E))^C \\
f^{-1}\left(\bigcup_{j=1}^{\infty} E_j\right) = \bigcup_{j=1}^{\infty} f^{-1}(E_j) \\
f^{-1}\left(\bigcap_{j=1}^{\infty} E_j\right) = \bigcap_{j=1}^{\infty} f^{-1}(E_j) \\
f^{-1}(\phi) = \phi\,,\ f^{-1}(Y) = X\,.
\end{gathered} \tag{3.3}$$

Putting these things together one sees that if $\mathcal{M}$ is any σ-algebra on X then

$$f_*(\mathcal{M}) = \left\{E \subset Y; f^{-1}(E) \in \mathcal{M}\right\} \tag{3.4}$$

is always a σ-algebra on Y.

In particular if $f^{-1}(A) \in \mathcal{M}$ for all $A \in G \subset \mathcal{N}$ then $f_*(\mathcal{M})$ is a σ-algebra containing G, hence containing $\mathcal{N}$ by the generating condition. Thus $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{N}$ so f is measurable. □

Proposition 3.2. *Any continuous map $f : X \to Y$ between metric spaces is measurable with respect to the Borel σ-algebras on X and Y.*

[4] Then X, or if you want to be pedantic $(X, \mathcal{M})$, is often said to be a *measure space* or even a *measurable space.*

Proof. The continuity of f shows that $f^{-1}(E) \subset X$ is open if $E \subset Y$ is open. By definition, the open sets generate the Borel σ-algebra on Y so the preceeding Lemma shows that f is Borel measurable i.e.,

$$f^{-1}(\mathcal{B}(Y)) \subset \mathcal{B}(X).$$

□

We are mainly interested in functions on X. If $\mathcal{M}$ is a σ-algebra on X then $f : X \to \mathbb{R}$ is *measurable* if it is measurable with respect to the Borel σ-algebra on $\mathbb{R}$ and $\mathcal{M}$ on X. More generally, for an extended function $f : X \to [-\infty, \infty]$ we take as the 'Borel' σ-algebra in $[-\infty, \infty]$ the smallest σ-algebra containing all open subsets of $\mathbb{R}$ and all sets $(a, \infty]$ and $[-\infty, b)$; in fact it is generated by the sets $(a, \infty]$. (See Problem 6.)

Our main task is to define the integral of a measurable function: we start with *simple functions*. Observe that the characteristic function of a set

$$\chi_E = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}$$

is measurable if and only if $E \in \mathcal{M}$. More generally a simple function,

$$f = \sum_{i=1}^{N} a_i \chi_{E_i}, \ a_i \in \mathbb{R} \tag{3.5}$$

is measurable if the E_i are measurable. The presentation, (3.5), of a simple function is not unique. We can make it so, getting the minimal presentation, by insisting that all the a_i are non-zero and

$$E_i = \{x \in E\,;\, f(x) = a_i\}$$

then f in (3.5) is measurable iff all the E_i are measurable.

The Lebesgue integral is based on approximation of functions by simple functions, so it is important to show that this is possible.

Proposition 3.3. *For any non-negative μ-measurable extended function $f : X \longrightarrow [0, \infty]$ there is an increasing sequence f_n of simple measurable functions such that $\lim_{n\to\infty} f_n(x) = f(x)$ for each $x \in X$ and this limit is uniform on any measurable set on which f is finite.*

Proof. Folland [1] page 45 has a nice proof. For each integer $n > 0$ and $0 \le k \le 2^{2n} - 1$, set

$$E_{n,k} = \{x \in X; 2^{-n}k \le f(x) < 2^{-n}(k+1)\},$$
$$E'_n = \{x \in X; f(x) \ge 2^n\}.$$

These are measurable sets. On increasing n by one, the interval in the definition of $E_{n,k}$ is divided into two. It follows that the sequence of simple functions

$$f_n = \sum_k 2^{-n} k \chi_{E_{k,n}} + 2^n \chi_{E'_n} \tag{3.6}$$

is increasing and has limit f and that this limit is uniform on any measurable set where f is finite. □