2. Measures and σ-algebras

An outer measure such as μ^* is a rather crude object since, even if the A_i are disjoint, there is generally strict inequality in (1.14). It turns out to be unreasonable to expect equality in (1.14), for disjoint unions, for a function defined on *all* subsets of X. We therefore restrict attention to smaller collections of subsets.

Definition 2.1. *A collection of subsets $\mathcal{M}$ of a set X is a σ-algebra if*

(1) $\phi, X \in \mathcal{M}$
(2) $E \in \mathcal{M} \Longrightarrow E^C = X \setminus E \in \mathcal{M}$
(3) $\{E_i\}_{i=1}^{\infty} \subset \mathcal{M} \Longrightarrow \bigcup_{i=1}^{\infty} E_i \in \mathcal{M}$.

For a general outer measure μ^* we define the notion of μ^*-measurability of a set.

Definition 2.2. *A set $E \subset X$ is μ^*-measurable (for an outer measure μ^* on X) if*

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^{\complement}) \ \forall \ A \subset X\,. \tag{2.1}$$

Proposition 2.3. *The collection of μ^*-measurable sets for any outer measure is a σ-algebra.*

Proof. Suppose E is μ^*-measurable, then E^C is μ^*-measurable by the symmetry of (2.1).

Suppose A, E and F are any three sets. Then

$$A \cap (E \cup F) = (A \cap E \cap F) \cup (A \cap E \cap F^C) \cup (A \cap E^C \cap F)$$
$$A \cap (E \cup F)^C = A \cap E^C \cap F^C\,.$$

From the subadditivity of μ^*

$$\begin{aligned}\mu^*(A \cap (E \cup F)) + \mu^*(A \cap (E \cup F)^C) \\ \le \mu^*(A \cap E \cap F) + \mu^*(A \cap E \cup F^C) \\ + \mu^*(A \cap E^C \cap F) + \mu^*(A \cap E^C \cap F^C).\end{aligned}$$

Now, if E and F are μ^*-measurable then applying the definition twice, for any A,

$$\begin{aligned}\mu^*(A) = \mu^*(A \cap E \cap F) + \mu^*(A \cap E \cap F^C) \\ + \mu^*(A \cap E^C \cap F) + \mu^*(A \cap E^C \cap F^C) \\ \ge \mu^*(A \cap (E \cup F)) + \mu^*(A \cap (E \cup F)^C)\,.\end{aligned}$$

The reverse inequality follows from the subadditivity of μ^*, so $E \cup F$ is also μ^*-measurable.

If $\{E_i\}_{i=1}^\infty$ is a sequence of disjoint μ^*-measurable sets, set $F_n = \bigcup_{i=1}^n E_i$ and $F = \bigcup_{i=1}^\infty E_i$. Then for any A,

$$\begin{aligned}\mu^*(A \cap F_n) &= \mu^*(A \cap F_n \cap E_n) + \mu^*(A \cap F_n \cap E_n^C)\\ &= \mu^*(A \cap E_n) + \mu^*(A \cap F_{n-1})\,.\end{aligned}$$

Iterating this shows that

$$\mu^*(A \cap F_n) = \sum_{j=1}^n \mu^*(A \cap E_j)\,.$$

From the μ^*-measurability of F_n and the subadditivity of μ^*,

$$\begin{aligned}\mu^*(A) &= \mu^*(A \cap F_n) + \mu^*(A \cap F_n^C)\\ &\geq \sum_{j=1}^n \mu^*(A \cap E_j) + \mu^*(A \cap F^C)\,.\end{aligned}$$

Taking the limit as $n \to \infty$ and using subadditivity,

$$\begin{aligned}\mu^*(A) &\geq \sum_{j=1}^\infty \mu^*(A \cap E_j) + \mu^*(A \cap F^C)\\ &\geq \mu^*(A \cap F) + \mu^*(A \cap F^C) \geq \mu^*(A)\end{aligned} \tag{2.2}$$

proves that inequalities are equalities, so F is also μ^*-measurable.

In general, for *any* countable union of μ^*-measurable sets,

$$\bigcup_{j=1}^\infty A_j = \bigcup_{j=1}^\infty \widetilde{A}_j\,,$$

$$\widetilde{A}_j = A_j \backslash \bigcup_{i=1}^{j-1} A_i = A_j \cap \left(\bigcup_{i=1}^{j-1} A_i\right)^C$$

is μ^*-measurable since the $\widetilde{A}_j$ are disjoint. □

A *measure* (sometimes called a *positive measure*) is an extended function defined on the elements of a σ-algebra $\mathcal{M}$:

$$\mu : \mathcal{M} \to [0, \infty]$$

such that

$$\mu(\emptyset) = 0 \text{ and} \tag{2.3}$$

$$\mu\left(\bigcup_{i=1}^\infty A_i\right) = \sum_{i=1}^\infty \mu(A_i) \tag{2.4}$$

if $\{A_i\}_{i=1}^\infty \subset \mathcal{M}$ and $A_i \cap A_j = \phi$ $i \neq j$.

The elements of $\mathcal{M}$ with measure zero, i.e., $E \in \mathcal{M}$, $\mu(E) = 0$, are supposed to be 'ignorable'. The measure μ is said to be *complete* if

$$E \subset X \text{ and } \exists\, F \in \mathcal{M}\,,\ \mu(F) = 0\,,\ E \subset F \Rightarrow E \in \mathcal{M}\,. \tag{2.5}$$

See Problem 4.

The first part of the following important result due to Caratheodory was shown above.

Theorem 2.4. *If μ^* is an outer measure on X then the collection of μ^*-measurable subsets of X is a σ-algebra and μ^* restricted to $\mathcal{M}$ is a complete measure.*

Proof. We have already shown that the collection of μ^*-measurable subsets of X is a σ-algebra. To see the second part, observe that taking $A = F$ in (2.2) gives

$$\mu^*(F) = \sum_j \mu^*(E_j) \text{ if } F = \bigcup_{j=1}^{\infty} E_j$$

and the E_j are disjoint elements of $\mathcal{M}$. This is (2.4).

Similarly if $\mu^*(E) = 0$ and $F \subset E$ then $\mu^*(F) = 0$. Thus it is enough to show that for any subset $E \subset X$, $\mu^*(E) = 0$ implies $E \in \mathcal{M}$. For any $A \subset X$, using the fact that $\mu^*(A \cap E) = 0$, and the 'increasing' property of μ^*

$$\begin{aligned} \mu^*(A) &\le \mu^*(A \cap E) + \mu^*(A \cap E^C) \\ &= \mu^*(A \cap E^C) \le \mu^*(A) \end{aligned}$$

shows that these must always be equalities, so $E \in \mathcal{M}$ (i.e., is μ^*-measurable). □

Going back to our primary concern, recall that we constructed the outer measure μ^* from $0 \le u \in (\mathcal{C}_0(X))'$ using (1.11) and (1.12). For the measure whose existence follows from Caratheodory's theorem to be much use we need

Proposition 2.5. *If $0 \le u \in (\mathcal{C}_0(X))'$, for X a locally compact metric space, then each open subset of X is μ^*-measurable for the outer measure defined by* (1.11) *and* (1.12) *and μ in* (1.11) *is its measure.*

Proof. Let $U \subset X$ be open. We only need to prove (2.1) for all $A \subset X$ with $\mu^*(A) < \infty$.[2]

[2] Why?

Suppose first that $A \subset X$ is open and $\mu^*(A) < \infty$. Then $A \cap U$ is open, so given $\epsilon > 0$ there exists $f \in C(X)$ $\mathrm{supp}(f) \Subset A \cap U$ with $0 \le f \le 1$ and

$$\mu^*(A \cap U) = \mu(A \cap U) \le u(f) + \epsilon\,.$$

Now, $A \setminus \mathrm{supp}(f)$ is also open, so we can find $g \in C(X)\,,\ 0 \le g \le 1\,,\ \mathrm{supp}(g) \Subset A \setminus \mathrm{supp}(f)$ with

$$\mu^*(A \setminus \mathrm{supp}(f)) = \mu(A \setminus \mathrm{supp}(f)) \le u(g) + \epsilon\,.$$

Since

$$A \setminus \mathrm{supp}(f) \supset A \cap U^C\,,\ 0 \le f + g \le 1\,,\ \mathrm{supp}(f+g) \Subset A\,,$$

$$\begin{aligned} \mu(A) &\ge u(f+g) = u(f) + u(g) \\ &> \mu^*(A \cap U) + \mu^*(A \cap U^C) - 2\epsilon \\ &\ge \mu^*(A) - 2\epsilon \end{aligned}$$

using subadditivity of μ^*. Letting $\epsilon \downarrow 0$ we conclude that

$$\mu^*(A) \le \mu^*(A \cap U) + \mu^*(A \cap U^C) \le \mu^*(A) = \mu(A)\,.$$

This gives (2.1) when A is open.

In general, if $E \subset X$ and $\mu^*(E) < \infty$ then given $\epsilon > 0$ there exists $A \subset X$ open with $\mu^*(E) > \mu^*(A) - \epsilon$. Thus,

$$\begin{aligned} \mu^*(E) &\ge \mu^*(A \cap U) + \mu^*(A \cap U^C) - \epsilon \\ &\ge \mu^*(E \cap U) + \mu^*(E \cap U^C) - \epsilon \\ &\ge \mu^*(E) - \epsilon\,. \end{aligned}$$

This shows that (2.1) always holds, so U is μ^*-measurable if it is open. We have already observed that $\mu(U) = \mu^*(U)$ if U is open. □

Thus we have shown that the σ-algebra given by Caratheodory's theorem contains all open sets. You showed in Problem 3 that the intersection of any collection of σ-algebras on a given set is a σ-algebra. Since $\mathcal{P}(X)$ is always a σ-algebra it follows that for *any* collection $\mathcal{E} \subset \mathcal{P}(X)$ there is always a smallest σ-algebra containing $\mathcal{E}$, namely

$$\mathcal{M}_{\mathcal{E}} = \bigcap \{\mathcal{M} \supset \mathcal{E}\,;\ \mathcal{M} \text{ is a } \sigma\text{-algebra}\,, \mathcal{M} \subset \mathcal{P}(X)\}\ .$$

The elements of the smallest σ-algebra containing the *open sets* are called 'Borel sets'. A measure defined on the σ-algebra of all Borel sets is called a *Borel measure*. This we have shown:

Proposition 2.6. *The measure defined by* (1.11), (1.12) *from* $0 \le u \in (\mathcal{C}_0(X))'$ *by Caratheodory's theorem is a Borel measure.*

Proof. This is what Proposition 2.5 says! See how easy proofs are. □

We can even continue in the same vein. A Borel measure is said to be *outer regular* on $E \subset X$ if

$$\mu(E) = \inf\{\mu(U)\,;\, U \supset E\,,\, U \text{ open}\}\ . \tag{2.6}$$

Thus the measure constructed in Proposition 2.5 is outer regular on all Borel sets! A Borel measure is *inner regular* on E if

$$\mu(E) = \sup\{\mu(K)\,;\, K \subset E\,,\, K \text{ compact}\}\ . \tag{2.7}$$

Here we need to know that compact sets are Borel measurable. This is Problem 5.

Definition 2.7. *A* Radon measure *(on a metric space) is a Borel measure which is outer regular on all Borel sets, inner regular on open sets and finite on compact sets.*

Proposition 2.8. *The measure defined by* (1.11), (1.12) *from* $0 \le u \in (\mathcal{C}_0(X))'$ *using Caratheodory's theorem is a Radon measure.*

Proof. Suppose $K \subset X$ is compact. Let χ_K be the characteristic function of K, $\chi_K = 1$ on K, $\chi_K = 0$ on K^C. Suppose $f \in \mathcal{C}_0(X)$, $\operatorname{supp}(f) \Subset X$ and $f \ge \chi_K$. Set

$$U_\epsilon = \{x \in X\,;\, f(x) > 1 - \epsilon\}$$

where $\epsilon > 0$ is small. Thus U_ϵ is open, by the continuity of f and contains K. Moreover, we can choose $g \in C(X)$, $\operatorname{supp}(g) \Subset U_\epsilon$, $0 \le g \le 1$ with $g = 1$ near[3] K. Thus, $g \le (1-\epsilon)^{-1} f$ and hence

$$\mu^*(K) \le u(g) = (1-\epsilon)^{-1} u(f)\,.$$

Letting $\epsilon \downarrow 0$, and using the measurability of K,

$$\mu(K) \le u(f)$$
$$\Rightarrow \mu(K) = \inf\{u(f)\,;\, f \in C(X)\,,\, \operatorname{supp}(f) \Subset X\ , f \ge \chi_K\}\ .$$

In particular this implies that $\mu(K) < \infty$ if $K \Subset X$, but is also proves (2.7). □

Let me now review a little of what we have done. We used the positive functional u to define an outer measure μ^*, hence a measure μ and then checked the properties of the latter.

This is a pretty nice scheme; getting ahead of myself a little, let me suggest that we try it on something else.

[3]Meaning in a neighborhood of K.

Let us say that $Q \subset \mathbb{R}^n$ is 'rectangular' if it is a product of finite intervals (open, closed or half-open)

$$Q = \prod_{i=1}^{n} (\text{or}[a_i\,,\,b_i]\text{or})\ a_i \le b_i \tag{2.8}$$

we all agree on its standard volume:

$$v(Q) = \prod_{i=1}^{n} (b_i - a_i) \in [0, \infty)\,. \tag{2.9}$$

Clearly if we have two such sets, $Q_1 \subset Q_2$, then $v(Q_1) \le v(Q_2)$. Let us try to define an outer measure on subsets of $\mathbb{R}^n$ by

$$v^*(A) = \inf \left\{ \sum_{i=1}^{\infty} v(Q_i)\,;\, A \subset \bigcup_{i=1}^{\infty} Q_i,\ Q_i \text{ rectangular} \right\}\,. \tag{2.10}$$

We want to show that (2.10) does define an outer measure. This is pretty easy; certainly $v(\emptyset) = 0$. Similarly if $\{A_i\}_{i=1}^{\infty}$ are (disjoint) sets and $\{Q_{ij}\}_{i=1}^{\infty}$ is a covering of A_i by open rectangles then all the Q_{ij} together cover $A = \bigcup_i A_i$ and

$$v^*(A) \le \sum_i \sum_j v(Q_{ij})$$

$$\Rightarrow v^*(A) \le \sum_i v^*(A_i)\,.$$

So we have an outer measure. We also want

Lemma 2.9. *If Q is rectangular then $v^*(Q) = v(Q)$.*

Assuming this, the measure defined from v^* using Caratheodory's theorem is called Lebesgue measure.

Proposition 2.10. *Lebesgue measure is a Borel measure.*

To prove this we just need to show that (open) rectangular sets are v^*-measurable.