16. Spectral theorem

For a bounded operator T on a Hilbert space we define the spectrum as the set

$$\operatorname{spec}(T) = \{z \in \mathbb{C}; T - z\operatorname{Id} \text{ is not invertible}\}. \tag{16.1}$$

Proposition 16.1. *For any bounded linear operator on a Hilbert space* $\operatorname{spec}(T) \subset \mathbb{C}$ *is a compact subset of* $\{|z| \le \|T\|\}$.

Proof. We show that the set $\mathbb{C}\setminus\operatorname{spec}(T)$ (generally called the resolvent set of T) is open and contains the complement of a sufficiently large ball. This is based on the convergence of the Neumann series. Namely if T is bounded and $\|T\| < 1$ then

$$(\operatorname{Id} - T)^{-1} = \sum_{j=0}^{\infty} T^j \tag{16.2}$$

converges to a bounded operator which is a two-sided inverse of $\operatorname{Id} - T$. Indeed, $\|T^j\| \le \|T\|^j$ so the series is convergent and composing with $\operatorname{Id} - T$ on either side gives a telescoping series reducing to the identity.

Applying this result, we first see that

$$(T - z) = -z(\operatorname{Id} - T/z) \tag{16.3}$$

is invertible if $|z| > \|T\|$. Similarly, if $(T - z_0)^{-1}$ exists for some $z_0 \in \mathbb{C}$ then

$$(T-z) = (T-z_0)-(z-z_0) = (T-z_0)^{-1}(\operatorname{Id} - (z-z_0)(T-z_0)^{-1}) \tag{16.4}$$

exists for $|z - z_0|\|(T - z_0)^{-1}\| < 1$. □

In general it is rather difficult to precisely locate $\operatorname{spec}(T)$.

However for a bounded self-adjoint operator it is easier. One sign of this is the the norm of the operator has an alternative, simple, characterization. Namely

$$\text{if } A^* = A \text{ then } \sup_{\|\phi\|=1} \langle A\phi, \phi\rangle| = \|A\|. \tag{16.5}$$

If a is this supermum, then clearly $a \le \|A\|$. To see the converse, choose any ϕ, $\psi \in H$ with norm 1 and then replace ψ by $e^{i\theta}\psi$ with θ chosen so that $\langle A\phi, \psi\rangle$ is real. Then use the polarization identity to write

$$\begin{aligned} 4\langle A\phi, \psi\rangle = {} & \langle A(\phi + \psi), (\phi + \psi)\rangle - \langle A(\phi - \psi), (\phi - \psi)\rangle \\ & + i\langle A(\phi + i\psi), (\phi + i\psi)\rangle - i\langle A(\phi - i\psi), (\phi - i\psi)\rangle. \end{aligned} \tag{16.6}$$

Now, by the assumed reality we may drop the last two terms and see that

$$4|\langle A\phi, \psi\rangle| \le a(\|\phi + \psi\|^2 + \|\phi - \psi\|^2) = 2a(\|\phi\|^2 + \|\psi\|^2) = 4a. \tag{16.7}$$

Thus indeed $\|A\| = \sup_{\|\phi\|=\|\psi\|=1} |\langle A\phi, \psi\rangle| = a.$

We can always subtract a real constant from A so that $A' = A - t$ satisfies

$$- \inf_{\|\phi\|=1} \langle A'\phi, \phi\rangle = \sup_{\|\phi\|=1} \langle A'\phi, \phi\rangle = \|A'\|. \tag{16.8}$$

Then, it follows that $A' \pm \|A'\|$ is not invertible. Indeed, there exists a sequence ϕ_n, with $\|\phi_n\| = 1$ such that $\langle (A' - \|A'\|)\phi_n, \phi_n\rangle \to 0$. Thus

$$\|(A' - \|A'\|)\phi_n\|^2 = -2\langle A'\phi_n, \phi_n\rangle + \|A'\phi_n\|^2 + \|A'\|^2 \le -2\langle A'\phi_n, \phi_n\rangle + 2\|A'\|^2 \to 0. \tag{16.9}$$

This shows that $A' - \|A'\|$ cannot be invertible and the same argument works for $A' + \|A'\|$. For the original operator A if we set

$$m = \inf_{\|\phi\|=1} \langle A\phi, \phi\rangle \ M = \sup_{\|\phi\|=1} \langle A\phi, \phi\rangle \tag{16.10}$$

then we conclude that neither $A - m\,\mathrm{Id}$ nor $A - M\,\mathrm{Id}$ is invertible and $\|A\| = \max(-m, M)$.

Proposition 16.2. *If A is a bounded self-adjoint operator then, with m and M defined by* (16.10)*,*

$$\{m\} \cup \{M\} \subset \operatorname{spec}(A) \subset [m, M]. \tag{16.11}$$

Proof. We have already shown the first part, that m and M are in the spectrum so it remains to show that $A - z$ is invertible for all $z \in \mathbb{C} \setminus [m, M]$.

Using the self-adjointness

$$\operatorname{Im}\langle (A - z)\phi, \phi\rangle = -\operatorname{Im} z\|\phi\|^2. \tag{16.12}$$

This implies that $A - z$ is invertible if $z \in \mathbb{C} \setminus \mathbb{R}$. First it shows that $(A - z)\phi = 0$ implies $\phi = 0$, so $A - z$ is injective. Secondly, the range is closed. Indeed, if $(A - z)\phi_n \to \psi$ then applying (16.12) directly shows that $\|\phi_n\|$ is bounded and so can be replaced by a weakly convergent subsequence. Applying (16.12) again to $\phi_n - \phi_m$ shows that the sequence is actually Cauchy, hence convergens to ϕ so $(A - z)\phi = \psi$ is in the range. Finally, the orthocomplement to this range is the null space of $A^* - \bar{z}$, which is also trivial, so $A - z$ is an isomorphism and (16.12) also shows that the inverse is bounded, in fact

$$\|(A - z)^{-1}\| \le \frac{1}{|\operatorname{Im} z|}. \tag{16.13}$$

When $z \in \mathbb{R}$ we can replace A by A' satisfying (16.8). Then we have to show that $A' - z$ is inverible for $|z| > \|A\|$, but that is shown in the proof of Proposition 16.1. $\square$

The basic estimate leading to the spectral theorem is:

Proposition 16.3. *If A is a bounded self-adjoint operator and p is a real polynomial in one variable,*

$$p(t) = \sum_{i=0}^{N} c_i t^i,\ c_N \neq 0, \tag{16.14}$$

then $p(A) = \sum_{i=0}^{N} c_i A^i$ *satisfies*

$$\|p(A)\| \leq \sup_{t \in [m,M]} |p(t)|. \tag{16.15}$$

Proof. Clearly, $p(A)$ is a bounded self-adjoint operator. If $s \notin p([m,M])$ then $p(A) - s$ is invertible. Indeed, the roots of $p(t) - s$ must cannot lie in $[m.M]$, since otherwise $s \in p([m,M])$. Thus, factorizing $p(s) - t$ we have

(16.16)

$$p(t) - s = c_N \prod_{i=1}^{N} (t - t_i(s)),\ t_i(s) \notin [m,M] \Longrightarrow (p(A) - s)^{-1} \text{ exists}$$

since $p(A) = c_N \sum_i (A - t_i(s))$ and each of the factors is invertible. Thus $\operatorname{spec}(p(A)) \subset p([m,M])$, which is an interval (or a point), and from Proposition 16.3 we conclude that $\|p(A)\| \leq \sup p([m,M])$ which is (16.15). □

Now, reinterpreting (16.15) we have a linear map

$$\mathcal{P}(\mathbb{R}) \ni p \longmapsto p(A) \in \mathcal{B}(H) \tag{16.17}$$

from the real polynomials to the bounded self-adjoint operators which is continuous with respect to the supremum norm on $[m,M]$. Since polynomials are dense in continuous functions on finite intervals, we see that (16.17) extends by continuity to a linear map

(16.18)

$$\mathcal{C}([m,M]) \ni f \longmapsto f(A) \in \mathcal{B}(H),\ \|f(A)\| \leq \|f\|_{[m,M]},\ fg(A) = f(A)g(A)$$

where the multiplicativity follows by continuity together with the fact that it is true for polynomials.

Now, consider any two elements $\phi, \psi \in H$. Evaluating $f(A)$ on ϕ and pairing with ψ gives a linear map

$$\mathcal{C}([m,M]) \ni f \longmapsto \langle f(A)\phi, \psi \rangle \in \mathbb{C}. \tag{16.19}$$

This is a linear functional on $\mathcal{C}([m,M])$ to which we can apply the Riesz representatin theorem and conclude that it is defined by integration against a unique Radon measure $\mu_{\phi,\psi}$:

$$\langle f(A)\phi,\psi\rangle = \int_{[m,M]} f d\mu_{\phi,\psi}. \tag{16.20}$$

The total mass $|\mu_{\phi,\psi}|$ of this measure is the norm of the functional. Since it is a Borel measure, we can take the integral on $-\infty, b]$ for any $b \in \mathbb{R}$ ad, with the uniqueness, this shows that we have a continuous sesquilinear map

(16.21)

$$P_b(\phi,\psi) : H\times H \ni (\phi,\psi) \longmapsto \int_{[m,b]} d\mu_{\phi,\psi} \in \mathbb{R},\ |P_b(\phi,\psi)| \le \|A\|\|\phi\|\|\psi\|.$$

From the Hilbert space Riesz representation theorem it follows that this sesquilinear form defines, and is determined by, a bounded linear operator

$$P_b(\phi,\psi) = \langle P_b\phi,\psi\rangle,\ \|P_b\| \le \|A\|. \tag{16.22}$$

In fact, from the functional calculus (the multiplicativity in (16.18)) we see that

$$P_b^* = P_b,\ P_b^2 = P_b,\ \|P_b\| \le 1, \tag{16.23}$$

so P_b is a projection.

Thus the spectral theorem gives us an increasing (with b) family of commuting self-adjoint projections such that $\mu_{\phi,\psi}((-\infty,b]) = \langle P_b\phi,\psi\rangle$ determines the Radon measure for which (16.20) holds. One can go further and think of P_b itself as determining a measure

$$\mu((-\infty,b]) = P_b \tag{16.24}$$

which takes values in the projections on H and which allows the functions of A to be written as integrals in the form

$$f(A) = \int_{[m,M]} f d\mu \tag{16.25}$$

of which (16.20) becomes the 'weak form'. To do so one needs to develop the theory of such measures and the corresponding integrals. This is not so hard but I shall not do it.