12. Cone support and wavefront set

In discussing the singular support of a tempered distibution above, notice that

$$\operatorname{singsupp}(u) = \emptyset$$

only implies that $u \in \mathcal{C}^\infty(\mathbb{R}^n)$, not as one might want, that $u \in \mathcal{S}(\mathbb{R}^n)$. We can however 'refine' the concept of singular support a little to get this.

Let us think of the sphere $\mathbb{S}^{n-1}$ as the set of 'asymptotic directions' in $\mathbb{R}^n$. That is, we identify a point in $\mathbb{S}^{n-1}$ with a half-line $\{a\bar{x}; a \in (0, \infty)\}$ for $0 \neq \bar{x} \in \mathbb{R}^n$. Since two points give the same half-line if and only if they are positive multiples of each other, this means we think of the sphere as the quotient

$$\mathbb{S}^{n-1} = (\mathbb{R}^n \setminus \{0\})/\mathbb{R}^+. \tag{12.1}$$

Of course if we have a metric on $\mathbb{R}^n$, for instance the usual Euclidean metric, then we can identify $\mathbb{S}^{n-1}$ with the unit sphere. However (12.1) does not require a choice of metric.

Now, suppose we consider functions on $\mathbb{R}^n \setminus \{0\}$ which are (positively) homogeneous of degree 0. That is $f(a\bar{x}) = f(\bar{x})$, for all $a > 0$, and they are just functions on $\mathbb{S}^{n-1}$. Smooth functions on $\mathbb{S}^{n-1}$ correspond (if you like by definition) with smooth functions on $\mathbb{R}^n \setminus \{0\}$ which are homogeneous of degree 0. Let us take such a function $\psi \in \mathcal{C}^\infty(\mathbb{R}^n \setminus \{0\})$, $\psi(ax) = \psi(x)$ for all $a > 0$. Now, to make this smooth on $\mathbb{R}^n$ we need to cut it off near 0. So choose a cutoff function $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, with $\chi(x) = 1$ in $|x| < 1$. Then

$$\psi_R(x) = \psi(x)(1 - \chi(x/R)) \in \mathcal{C}^\infty(\mathbb{R}^n), \tag{12.2}$$

for any $R > 0$. This function is supported in $|x| \geq R$. Now, if ψ has support near some point $\omega \in \mathbb{S}^{n-1}$ then for R large the corresponding function ψ_R will 'localize near ω as a point at infinity of $\mathbb{R}^n$.' Rather than try to understand this directly, let us consider a corresponding analytic construction.

First of all, a function of the form ψ_R is a multiplier on $\mathcal{S}(\mathbb{R}^n)$. That is,

$$\psi_R \cdot : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n). \tag{12.3}$$

To see this, the main problem is to estimate the derivatives at infinity, since the product of smooth functions is smooth. This in turn amounts to estimating the deriviatives of ψ in $|x| \geq 1$. This we can do using the homogeneity.

Lemma 12.1. *If* $\psi \in \mathcal{C}^\infty(\mathbb{R}^n \setminus \{0\})$ *is homogeneous of degree* 0 *then*

$$|D^\alpha \psi| \le C_\alpha |x|^{-|\alpha|}. \tag{12.4}$$

Proof. I should not have even called this a lemma. By the chain rule, the derivative of order α is a homogeneous function of degree $-|\alpha|$ from which (12.4) follows. □

For the smoothed versio, ψ_R, of ψ this gives the estimates

$$|D^\alpha \psi_R(x)| \le C_\alpha \langle x\rangle^{-|\alpha|}. \tag{12.5}$$

This allows us to estimate the derivatives of the product of a Schwartz function and ψ_R :

$$\begin{aligned}(12.6)\quad & x^\beta D^\alpha(\psi_R f) \\ &= \sum_{\gamma\le\alpha}\binom{\alpha}{\gamma} D^{\alpha-\gamma}\psi_R x^\beta D^\gamma f \Longrightarrow \sup_{|x|\ge 1}|x^\beta D^\alpha(\psi_R f)| \le C \sup \|f\|_k\end{aligned}$$

for some seminorm on $\mathcal{S}(\mathbb{R}^n)$. Thus the map (12.3) is actually continuous. This continuity means that ψ_R is a multiplier on $\mathcal{S}'(\mathbb{R}^n)$, defined as usual by duality:

$$\psi_R u(f) = u(\psi_R f)\ \forall\ f \in \mathcal{S}(\mathbb{R}^n). \tag{12.7}$$

Definition 12.2. *The cone-support and cone-singular-support of a tempered distribution are the subsets* $\operatorname{Csp}(u) \subset \mathbb{R}^n \cup \mathbb{S}^{n-1}$ *and* $\operatorname{Css}(u) \subset \mathbb{R}^n \cup \mathbb{S}^{n-1}$ *defined by the conditions*

(12.8)
$$\begin{aligned}\operatorname{Csp}(u)\cap\mathbb{R}^n &= \operatorname{supp}(u)\\ (\operatorname{Csp}(u))^{\complement}\cap\mathbb{S}^{n-1} &= \{\omega\in\mathbb{S}^{n-1};\\ &\qquad \exists\ R>0,\ \psi\in\mathcal{C}^\infty(\mathbb{S}^{n-1}),\ \psi(\omega)\ne 0,\ \psi_R u = 0\},\\ \operatorname{Css}(u)\cap\mathbb{R}^n &= \operatorname{singsupp}(u)\\ (\operatorname{Css}(u))^{\complement}\cap\mathbb{S}^{n-1} &= \{\omega\in\mathbb{S}^{n-1};\\ &\qquad \exists\ R>0,\ \psi\in\mathcal{C}^\infty(\mathbb{S}^{n-1}),\ \psi(\omega)\ne 0,\ \psi_R u\in\mathcal{S}(\mathbb{R}^n)\}.\end{aligned}$$

That is, on the $\mathbb{R}^n$ part these are the same sets as before but 'at infinity' they are defined by conic localization on $\mathbb{S}^{n-1}$.

In considering $\operatorname{Csp}(u)$ and $\operatorname{Css}(u)$ it is convenient to combine $\mathbb{R}^n$ and $\mathbb{S}^{n-1}$ into a compactification of $\mathbb{R}^n$. To do so (topologically) let us identify $\mathbb{R}^n$ with the interior of the unit ball with respect to the Euclidean metric using the map

$$\mathbb{R}^n \ni x \longmapsto \frac{x}{\langle x\rangle} \in \{y\in\mathbb{R}^n; |y|\le 1\} = \mathbb{B}^n. \tag{12.9}$$

Clearly $|x| < \langle x\rangle$ and for $0 \le a < 1$, $|x| = a\langle x\rangle$ has only the solution $|x| = a/(1-a^2)^{\frac12}$. Thus if we combine (12.9) with the identification of $\mathbb{S}^n$ with the unit sphere we get an identification

$$\mathbb{R}^n \cup \mathbb{S}^{n-1} \simeq \mathbb{B}^n. \tag{12.10}$$

Using this identification we can, and will, regard $\mathrm{Csp}(u)$ and $\mathrm{Css}(u)$ as subsets of $\mathbb{B}^n$.[21]

Lemma 12.3. *For any* $u \in \mathcal{S}'(\mathbb{R}^n)$, $\mathrm{Csp}(u)$ *and* $\mathrm{Css}(u)$ *are closed subsets of* $\mathbb{B}^n$ *and if* $\tilde\psi \in \mathcal{C}^\infty(\mathbb{S}^n)$ *has* $\mathrm{supp}(\tilde\psi) \cap \mathrm{Css}(u) = \emptyset$ *then for* R *sufficiently large* $\tilde\psi_R u \in \mathcal{S}(\mathbb{R}^n)$.

Proof. Directly from the definition we know that $\mathrm{Csp}(u) \cap \mathbb{R}^n$ is closed, as is $\mathrm{Css}(u) \cap \mathbb{R}^n$. Thus, in each case, we need to show that if $\omega \in \mathbb{S}^{n-1}$ and $\omega \notin \mathrm{Csp}(u)$ then $\mathrm{Csp}(u)$ is disjoint from some neighbourhood of ω in $\mathbb{B}^n$. However, by definition,

$$U = \{x \in \mathbb{R}^n; \psi_R(x) \ne 0\} \cup \{\omega' \in \mathbb{S}^{n-1}; \psi(\omega') \ne 0\}$$

is such a neighbourhood. Thus the fact that $\mathrm{Csp}(u)$ is closed follows directly from the definition. The argument for $\mathrm{Css}(u)$ is essentially the same.

The second result follows by the use of a partition of unity on $\mathbb{S}^{n-1}$. Thus, for each point in $\mathrm{supp}(\psi) \subset \mathbb{S}^{n-1}$ there exists a conic localizer for which $\psi_R u \in \mathcal{S}(\mathbb{R}^n)$. By compactness we may choose a finite number of these functions ψ_j such that the open sets $\{\psi_j(\omega) > 0\}$ cover $\mathrm{supp}(\tilde\psi)$. By assumption $(\psi_j)_{R_j} u \in \mathcal{S}(\mathbb{R}^n)$ for some $R_j > 0$. However this will remain true if R_j is increased, so we may suppose that $R_j = R$ is independent of j. Then for function

$$\mu = \sum_j |\psi_j|^2 \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$$

we have $\mu_R u \in \mathcal{S}(\mathbb{R}^n)$. Since $\tilde\psi = \psi'\mu$ for some $\mu \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$ it follows that $\tilde\psi_{R+1} u \in \mathcal{S}(\mathbb{R}^n)$ as claimed. □

Corollary 12.4. *If* $u \in \mathcal{S}'(\mathbb{R}^n)$ *then* $\mathrm{Css}(u) = \emptyset$ *if and only if* $u \in \mathcal{S}(\mathbb{R}^n)$.

Proof. Certainly $\mathrm{Css}(u) = \emptyset$ if $u \in \mathcal{S}(\mathbb{R}^n)$. If $u \in \mathcal{S}'(\mathbb{R}^n)$ and $\mathrm{Css}(u) = \emptyset$ then from Lemma 12.3, $\psi_R u \in \mathcal{S}(\mathbb{R}^n)$ where $\psi = 1$. Thus $v = (1-\psi_R)u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ has $\mathrm{singsupp}(v) = \emptyset$ so $v \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ and hence $u \in \mathcal{S}(\mathbb{R}^n)$. □

[21] In fact while the topology here is correct the smooth structure on $\mathbb{B}^n$ is *not the right one*™– see Problem?? For our purposes here this issue is irrelevant.

Of course the analogous result for $\mathrm{Csp}(u)$, that $\mathrm{Csp}(u) = \emptyset$ if and only if $u = 0$ follows from the fact that this is true if $\mathrm{supp}(u) = \emptyset$. I will treat a few other properties as self-evident. For instance

$$\mathrm{Csp}(\phi u) \subset \mathrm{Csp}(u),\ \mathrm{Css}(\phi u) \subset \mathrm{Css}(u)\ \forall\ u \in \mathcal{S}'(\mathbb{R}^n),\ \phi \in \mathcal{S}(\mathbb{R}^n) \tag{12.11}$$

and

$$\begin{gathered}\mathrm{Csp}(c_1u_1 + c_2u_2) \subset \mathrm{Csp}(u_1) \cup \mathrm{Csp}(u_2),\\ \mathrm{Css}(c_1u_1 + c_2u_2) \subset \mathrm{Css}(u_1) \cup \mathrm{Css}(u_2)\\ \forall\ u_1, u_2 \in \mathcal{S}'(\mathbb{R}^n),\ c_1, c_2 \in \mathbb{C}.\end{gathered} \tag{12.12}$$

One useful consequence of having the cone support at our disposal is that we can discuss sufficient conditions to allow us to multiply distributions; we will get better conditions below using the same idea but applied to the wavefront set but this preliminary discussion is used there. In general the product of two distributions is not defined, and indeed not definable, as a distribution. However, we can always multiply an element of $\mathcal{S}'(\mathbb{R}^n)$ and an element of $\mathcal{S}(\mathbb{R}^n)$.

To try to understand multiplication look at the question of *pairing* between two distributions.

Lemma 12.5. *If $K_i \subset \mathbb{B}^n$, $i = 1, 2$, are two disjoint closed (hence compact) subsets then we can define an unambiguous pairing*

$$\begin{gathered}\{u \in \mathcal{S}'(\mathbb{R}^n); \mathrm{Css}(u) \subset K_1\} \times \{u \in \mathcal{S}'(\mathbb{R}^n); \mathrm{Css}(u) \subset K_2\} \ni (u_1, u_2)\\ \longrightarrow u_1(u_2) \in \mathbb{C}.\end{gathered} \tag{12.13}$$

Proof. To define the pairing, choose a function $\psi \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$ which is identically equal to 1 in a neighbourhood of $K_1 \cap \mathbb{S}^{n-1}$ and with support disjoint from $K_2 \cap \mathbb{S}^{n-1}$. Then extend it to be homogeneous, as above, and cut off to get ψ_R. If R is large enough $\mathrm{Csp}(\psi_R)$ is disjoint from K_2. Then $\psi_R + (1 - \psi)_R = 1 + \nu$ where $\nu \in \mathcal{C}_c^\infty(\mathbb{R}^n)$. We can find another function $\mu \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ such that $\psi_1 = \psi_R + \mu = 1$ in a neighbourhood of K_1 and with $\mathrm{Csp}(\psi_1)$ disjoint from K_2. Once we have this, for u_1 and u_2 as in (12.13),

$$\psi_1 u_2 \in \mathcal{S}(\mathbb{R}^n) \text{ and } (1 - \psi_1)u_1 \in \mathcal{S}(\mathbb{R}^n) \tag{12.14}$$

since in both cases Css is empty from the definition. Thus we can define the desired pairing between u_1 and u_2 by

$$u_1(u_2) = u_1(\psi_1 u_2) + u_2((1 - \psi_1)u_1). \tag{12.15}$$

Of course we should check that this definition is independent of the cut-off function used in it. However, if we go through the definition and choose a different function ψ' to start with, extend it homogeneoulsy and cut off (probably at a different R) and then find a correction term μ' then the 1-parameter linear homotopy between them

$$\psi_1(t) = t\psi_1 + (1-t)\psi_1',\ t \in [0,1] \tag{12.16}$$

satisfies all the conditions required of ψ_1 in formula (12.14). Thus in fact we get a smooth family of pairings, which we can write for the moment as

$$(u_1, u_2)_t = u_1(\psi_1(t)u_2) + u_2((1-\psi_1(t))u_1). \tag{12.17}$$

By inspection, this is an affine-linear function of t with derivative

$$u_1((\psi_1 - \psi_1')u_2) + u_2((\psi_1' - \psi_1))u_1). \tag{12.18}$$

Now, we just have to justify moving the smooth function in (12.18) to see that this gives zero. This should be possible since $\mathrm{Csp}(\psi_1' - \psi_1)$ is disjoint from *both* K_1 and K_2.

In fact, to be very careful for once, we should construct another function χ in the same way as we constructed ψ_1 to be homogenous near infinity and smooth and such that $\mathrm{Csp}(\chi)$ is also disjoint from both K_1 and K_2 but $\chi = 1$ on $\mathrm{Csp}(\psi_1' - \psi_1)$. Then $\chi(\psi_1' - \psi_1) = \psi_1' - \psi_1$ so we can insert it in (12.18) and justify

$$\begin{aligned} u_1((\psi_1 - \psi_1')u_2) &= u_1(\chi^2(\psi_1 - \psi_1')u_2) = (\chi u_1)((\psi_1 - \psi_1')\chi u_2) \\ &= (\chi u_2)(\psi_1 - \psi_1')\chi u_1) = u_2(\psi_1 - \psi_1')\chi u_1). \end{aligned} \tag{12.19}$$

Here the second equality is just the identity for χ as a (multiplicative) linear map on $\mathcal{S}(\mathbb{R}^n)$ and hence $\mathcal{S}'(\mathbb{R}^n)$ and the operation to give the crucial, third, equality is permissible because both elements are in $\mathcal{S}(\mathbb{R}^n)$. □

Once we have defined the pairing between tempered distibutions with disjoint conic singular supports, in the sense of (12.14), (12.15), we can define the product under the same conditions. Namely to define the product of say u_1 and u_2 we simply set

$$\begin{aligned} &u_1u_2(\phi) = u_1(\phi u_2) = u_2(\phi u_1)\ \forall\ \phi \in \mathcal{S}(\mathbb{R}^n), \\ &\qquad\qquad \text{provided } \mathrm{Css}(u_1) \cap \mathrm{Css}(u_2) = \emptyset. \end{aligned} \tag{12.20}$$

Indeed, this would be true if one of u_1 or u_2 was itself in $\mathcal{S}(\mathbb{R}^n)$ and makes sense in general. I leave it to you to check the continuity statement required to prove that the product is actually a tempered distibution (Problem 78).

One can also give a similar discussion of the convolution of two tempered distributions. Once again we do not have a definition of $u*v$ as a tempered distribution for all u, $v \in \mathcal{S}'(\mathbb{R}^n)$. We do know how to define the convolution if either u or v is compactly supported, or if either is in $\mathcal{S}(\mathbb{R}^n)$. This leads directly to

Lemma 12.6. *If* $\mathrm{Css}(u) \cap \mathbb{S}^{n-1} = \emptyset$ *then* $u*v$ *is defined unambiguously by*

$$u * v = u_1 * v + u_2 * v,\ u_1 = (1 - \chi(\frac{x}{r}))u,\ u_2 = u - u_1 \tag{12.21}$$

where $\chi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ *has* $\chi(x) = 1$ *in* $|x| \le 1$ *and* R *is sufficiently large; there is a similar definition if* $\mathrm{Css}(v) \cap \mathbb{S}^{n-1} = \emptyset$.

Proof. Since $\mathrm{Css}(u) \cap \mathbb{S}^{n-1} = \emptyset$, we know that $\mathrm{Css}(u_1) = \emptyset$ if R is large enough, so then both terms on the right in (12.21) are well-defined. To see that the result is independent of R just observe that the difference of the right-hand side for two values of R is of the form $w * v - w * v$ with w compactly supported. □

Now, we can go even further using a slightly more sophisticated decomposition based on

Lemma 12.7. *If* $u \in \mathcal{S}'(\mathbb{R}^n)$ *and* $\mathrm{Css}(u) \cap \Gamma = \emptyset$ *where* $\Gamma \subset \mathbb{S}^{n-1}$ *is a closed set, then* $u = u_1 + u_2$ *where* $\mathrm{Csp}(u_1) \cap \Gamma = \emptyset$ *and* $u_2 \in \mathcal{S}(\mathbb{R}^n)$; *in fact*

$$\begin{gathered} u = u_1' + u_1'' + u_2 \text{ where } u_1' \in \mathcal{C}^{-\infty}_c(\mathbb{R}^n) \text{ and} \\ 0 \notin \mathrm{supp}(u_1''),\ x \in \mathbb{R}^n \setminus \{0\},\ x/|x| \in \Gamma \Longrightarrow x \notin \mathrm{supp}(u_1''). \end{gathered} \tag{12.22}$$

Proof. A covering argument which you should provide. □

Let $\Gamma_i \subset \mathbb{R}^n$, $i = 1, 2$, be closed cones. That is they are closed sets such that if $x \in \Gamma_i$ and $a > 0$ then $ax \in \Gamma_i$. Suppose in addition that

$$\Gamma_1 \cap (-\Gamma_2) = \{0\}. \tag{12.23}$$

That is, if $x \in \Gamma_1$ and $-x \in \Gamma_2$ then $x = 0$. Then it follows that for some $c > 0$,

$$x \in \Gamma_1,\ y \in \Gamma_2 \Longrightarrow |x + y| \ge c(|x| + |y|). \tag{12.24}$$

To see this consider $x + y$ where $x \in \Gamma_1$, $y \in \Gamma_2$ and $|y| \le |x|$. We can assume that $x \ne 0$, otherwise the estimate is trivially true with $c = 1$, and then $Y = y/|x| \in \Gamma_1$ and $X = x/|x| \in \Gamma_2$ have $|Y| \le 1$ and $|X| = 1$. However $X + Y \ne 0$, since $|X| = 1$, so by the continuity of the sum, $|X + Y| \ge 2c > 0$ for some $c > 0$. Thus $|X + Y| \ge c(|X| + |Y|)$ and the result follows by scaling back. The other case, of $|x| \le |y|$

follows by the same argument with x and y interchanged, so (12.24) is a consequence of (12.23).

Lemma 12.8. *For any* $u \in \mathcal{S}'(\mathbb{R}^n)$ *and* $\phi \in \mathcal{S}(\mathbb{R}^n)$,

$$\operatorname{Css}(\phi * u) \subset \operatorname{Css}(u) \cap \mathbb{S}^{n-1}. \tag{12.25}$$

Proof. We already know that $\phi*u$ is smooth, so $\operatorname{Css}(\phi*u) \subset \mathbb{S}^{n-1}$. Thus, we need to show that if $\omega \in \mathbb{S}^{n-1}$ and $\omega \notin \operatorname{Css}(u)$ then $\omega \notin \operatorname{Css}(\phi * u)$.

Fix such a point $\omega \in \mathbb{S}^{n-1} \setminus \operatorname{Css}(u)$ and take a closed set $\Gamma \subset \mathbb{S}^{n-1}$ which is a neighbourhood of ω but which is still disjoint from $\operatorname{Css}(u)$ and then apply Lemma 12.7. The two terms $\phi * u_2$, where $u_2 \in \mathcal{S}(\mathbb{R}^n)$ and $\phi * u_1'$ where $u_1' \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ are both in $\mathcal{S}(\mathbb{R}^n)$ so we can assume that u has the support properties of u_1''. In particular there is a smaller closed subset $\Gamma_1 \subset \mathbb{S}^{n-1}$ which is still a neighbourhood of ω but which does not meet Γ_2, which is the closure of the complement of Γ. If we replace these Γ_i by the closed cones of which they are the 'cross-sections' then we are in the situation of (12.23) and (12.23), except for the signs. That is, there is a constant $c > 0$ such that

$$|x - y| \ge c(|x| + |y|). \tag{12.26}$$

Now, we can assume that there is a cutoff function ψ_R which has support in Γ_2 and is such that $u = \psi_R u$. For any conic cutoff, ψ_R', with support in Γ_1

$$\psi_R'(\phi * u) = \langle \psi_R u, \phi(x - \cdot)\rangle = \langle u(y), \psi_R(y)\psi_R'(x)\phi(x - y)\rangle. \tag{12.27}$$

The continuity of u means that this is estimated by some Schwartz seminorm

$$\begin{aligned} &\sup_{y,|\alpha|\le k} |D_y^\alpha(\psi_R(y)\psi_R'(x)\phi(x-y))|(1+|y|)^k \\ &\le C_N\|\phi\| \sup_y (1+|x|+|y|)^{-N}(1+|y|)^k \le C_N\|\phi\|(1+|x|)^{-N+k} \end{aligned} \tag{12.28}$$

for some Schwartz seminorm on ϕ. Here we have used the estimate (12.24), in the form (12.26), using the properties of the supports of ψ_R' and ψ_R. Since this is true for any N and similar estimates hold for the derivatives, it follows that $\psi_R'(u * \phi) \in \mathcal{S}(\mathbb{R}^n)$ and hence that $\omega \notin \operatorname{Css}(u * \phi)$. □

Corollary 12.9. *Under the conditions of Lemma 12.6*

$$\operatorname{Css}(u * v) \subset (\operatorname{singsupp}(u) + \operatorname{singsupp}(v)) \cup (\operatorname{Css}(v) \cap \mathbb{S}^{n-1}). \tag{12.29}$$

Proof. We can apply Lemma 12.8 to the first term in (12.21) to conclude that it has conic singular support contained in the second term in (12.29). Thus it is enough to show that (12.29) holds when $u \in$

$\mathcal{C}_c^{-\infty}(\mathbb{R}^n)$. In that case we know that the singular support of the convolution is contained in the first term in (12.29), so it is enough to consider the conic singular support in the sphere at infinity. Thus, if $\omega \notin \operatorname{Css}(v)$ we need to show that $\omega \notin \operatorname{Css}(u * v)$. Using Lemma 12.7 we can decompose $v = v_1 + v_2 + v_3$ as a sum of a Schwartz term, a compact supported term and a term which does not have ω in its conic support. Then $u * v_1$ is Schwartz, $u * v_2$ has compact support and satisfies (12.29) and ω is not in the cone support of $u * v_3$. Thus (12.29) holds in general. □

Lemma 12.10. *If $u,\ v \in \mathcal{S}'(\mathbb{R}^n)$ and $\omega \in \operatorname{Css}(u) \cap \mathbb{S}^{n-1} \Longrightarrow -\omega \notin \operatorname{Css}(v)$ then their convolution is defined unambiguously, using the pairing in Lemma 12.5, by*

$$u * v(\phi) = u(\check{v} * \phi)\ \forall\ \phi \in \mathcal{S}(\mathbb{R}^n). \tag{12.30}$$

Proof. Since $\check{v}(x) = v(-x)$, $\operatorname{Css}(\check{v}) = -\operatorname{Css}(v)$ so applying Lemma 12.8 we know that

$$\operatorname{Css}(\check{v} * \phi) \subset -\operatorname{Css}(v) \cap \mathbb{S}^{n-1}. \tag{12.31}$$

Thus, $\operatorname{Css}(v) \cap \operatorname{Css}(\check{v} * \phi) = \emptyset$ and the pairing on the right in (12.30) is well-defined by Lemma 12.5. Continuity follows from your work in Problem 78. □

In Problem 79 I ask you to get a bound on $\operatorname{Css}(u * v) \cap \mathbb{S}^{n-1}$ under the conditions in Lemma 12.10.

Let me do what is actually a fundamental computation.

Lemma 12.11. *For a conic cutoff, ψ_R, where $\psi \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$,*

$$\operatorname{Css}(\widehat{\psi_R}) \subset \{0\}. \tag{12.32}$$

Proof. This is actually much easier than it seems. Namely we already know that $D^\alpha(\psi_R)$ is smooth and homogeneous of degree $-|\alpha|$ near infinity. From the same argument it follows that

$$D^\alpha(x^\beta \psi_R) \in L^2(\mathbb{R}^n) \text{ if } |\alpha| > |\beta| + n/2 \tag{12.33}$$

since this is a smooth function homogeneous of degree less than $-n/2$ near infinity, hence square-integrable. Now, taking the Fourier transform gives

$$\xi^\alpha D^\beta(\widehat{\psi_R}) \in L^2(\mathbb{R}^n)\ \forall\ |\alpha| > |\beta| + n/2. \tag{12.34}$$

If we localize in a cone near infinity, using a (completely unrelated) cutoff $\psi'_{R'}(\xi)$ then we must get a Schwartz function since

$$|\xi|^{|\alpha|}\psi'_{R'}(\xi) D^\beta(\widehat{\psi_R}) \in L^2(\mathbb{R}^n)\ \forall\ |\alpha| > |\beta| + n/2 \Longrightarrow \psi'_{R'}(\xi)\widehat{\psi_R} \in \mathcal{S}(\mathbb{R}^n). \tag{12.35}$$

Indeed this argument applies anywhere that $\xi \neq 0$ and so shows that (12.32) holds. □

Now, we have obtained some reasonable looking conditions under which the product uv or the convolution $u*v$ of two elements of $\mathcal{S}'(\mathbb{R}^n)$ is defined. However, reasonable as they might be there is clearly a flaw, or at least a deficiency, in the discussion. We know that in the simplest of cases,

$$\widehat{u*v} = \widehat{u}\widehat{v}. \tag{12.36}$$

Thus, it is very natural to expect a relationship between the conditions under which the product of the Fourier transforms is defined and the conditions under which the convolution is defined. Is there? Well, not much it would seem, since on the one hand we are considering the relationship between $\mathrm{Css}(\widehat{u})$ and $\mathrm{Css}(\widehat{v})$ and on the other the relationship between $\mathrm{Css}(u)\cap\mathbb{S}^{n-1}$ and $\mathrm{Css}(v)\cap\mathbb{S}^{n-1}$. If these are to be related, we would have to find a relationship of some sort between $\mathrm{Css}(u)$ and $\mathrm{Css}(\widehat{u})$. As we shall see, there is one but it is not very strong as can be guessed from Lemma 12.11. This is not so much a bad thing as a sign that we should look for another notion which combines aspects of both $\mathrm{Css}(u)$ and $\mathrm{Css}(\widehat{u})$. This we will do through the notion of *wave-front set*. In fact we define two related objects. The first is the more conventional, the second is more natural in our present discussion.

Definition 12.12. *If $u \in \mathcal{S}'(\mathbb{R}^n)$ we define the* wavefront set *of u to be*

$$\begin{aligned}\mathrm{WF}(u) = \{(x,\omega)\in\mathbb{R}^n\times\mathbb{S}^{n-1};\\ \exists\ \phi\in\mathcal{C}^\infty_c(\mathbb{R}^n),\ \phi(x)\neq 0,\ \omega\notin\mathrm{Css}(\widehat{\phi u})\}^{\complement}\end{aligned} \tag{12.37}$$

and more generally the scattering wavefront set *by*

$$\begin{aligned}\mathrm{WF}_{\mathrm{sc}}(u) = \mathrm{WF}(u)\cup\{(\omega,p)\in\mathbb{S}^{n-1}\times\mathbb{B}^n;\\ \exists\ \psi\in\mathcal{C}^\infty(\mathbb{S}^n),\ \psi(\omega)\neq 0,\ R>0 \text{ such that } p\notin\mathrm{Css}(\widehat{\psi_R u})\}^{\complement}.\end{aligned} \tag{12.38}$$

So, the definition is really always the same. To show that $(p,q)\notin \mathrm{WF}_{\mathrm{sc}}(u)$ we need to find 'a cutoff Φ near p' – depending on whether $p\in\mathbb{R}^n$ or $p\in\mathbb{S}^{n-1}$ this is either $\Phi=\phi\in\mathcal{C}^\infty_c(\mathbb{R}^n)$ with $F=\phi(p)\neq 0$ or a ψ_R where $\psi\in\mathcal{C}^\infty(\mathbb{S}^{n-1})$ has $\psi(p)\neq 0$ – such that $q\notin\mathrm{Css}(\widehat{\Phi u})$. One crucial property is

Lemma 12.13. *If $(p,q)\notin\mathrm{WF}_{\mathrm{sc}}(u)$ then if $p\in\mathbb{R}^n$ there exists a neighbourhood $U\subset\mathbb{R}^n$ of p and a neighbourhood $U\subset\mathbb{B}^n$ of q such that for all $\phi\in\mathcal{C}^\infty_c(\mathbb{R}^n)$ with support in U, $U'\cap\mathrm{Css}(\widehat{\phi u})=\emptyset$; similarly* *if* $p \in \mathbb{S}^{n-1}$ *then there exists a neigbourhood* $\tilde{U} \subset \mathbb{B}^n$ *of* p *such that* $U' \cap \operatorname{Css}(\widehat{\psi_R u}) = \emptyset$ *if* $\operatorname{Csp}(\omega_R) \subset \tilde{U}$.

Proof. First suppose $p \in \mathbb{R}^n$. From the definition of conic singular support, (12.37) means precisely that there exists $\psi \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$, $\psi(\omega) \neq 0$ and R such that

$$\psi_R(\widehat{\phi u}) \in \mathcal{S}(\mathbb{R}^n). \tag{12.39}$$

Since we know that $\widehat{\phi u} \in \mathcal{C}^\infty(\mathbb{R}^n)$, this is actually true for all $R > 0$ as soon as it is true for one value. Furthermore, if $\phi' \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ has $\operatorname{supp}(\phi') \subset \{\phi \neq 0\}$ then $\omega \notin \operatorname{Css}(\widehat{\phi' u})$ follows from $\omega \notin \operatorname{Css}(\widehat{\phi u})$. Indeed we can then write $\phi' = \mu\phi$ where $\mu \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ so it suffices to show that if $v \in \mathcal{C}^{-\infty}_c(\mathbb{R}^n)$ has $\omega \notin \operatorname{Css}(\widehat{v})$ then $\omega \notin \operatorname{Css}(\widehat{\mu v})$ if $\mu \in \mathcal{C}^\infty_c(\mathbb{R}^n)$. Since $\widehat{\mu v} = (2\pi)^{-n} v * \widehat{u}$ where $\check{v} = \widehat{\mu} \in \mathcal{S}(\mathbb{R}^n)$, applying Lemma 12.8 we see that $\operatorname{Css}(v * \widehat{v}) \subset \operatorname{Css}(\widehat{v})$, so indeed $\omega \notin \operatorname{Css}(\widehat{\phi' u})$.

The case that $p \in \mathbb{S}^{n-1}$ is similar. Namely we have one cut-off ψ_R with $\psi(p) \neq 0$ and $q \notin \operatorname{Css}(\widehat{\omega_R u})$. We can take $U = \{\psi_{R+10} \neq 0\}$ since if $\psi'_{R'}$ has conic support in U then $\psi'_{R'} = \psi'' R' \psi_R$ for some $\psi'' \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$. Thus

$$\widehat{\psi'_{R'} u} = v * \widehat{\psi_R u},\ \check{v} = \widehat{\omega''_{R''}}. \tag{12.40}$$

From Lemma 12.11 and Corollary12.9 we deduce that

$$\operatorname{Css}(\widehat{\psi'_{R'} u}) \subset \operatorname{Css}(\widehat{\omega_R u}) \tag{12.41}$$

and hence the result follows with U' a small neighourhood of q. □

Proposition 12.14. *For any* $u \in \mathcal{S}'(\mathbb{R}^n)$,

$$\begin{aligned} \operatorname{WF}_{\mathrm{sc}}(u) \subset \partial(\mathbb{B}^n \times \mathbb{B}^n) &= (\mathbb{B}^n \times \mathbb{S}^{n-1}) \cup (\mathbb{S}^{n-1} \times \mathbb{B}^n) \\ &= (\mathbb{R}^n \times \mathbb{S}^{n-1}) \cup (\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}) \cup (\mathbb{S}^{n-1} \times \mathbb{R}^n) \end{aligned} \tag{12.42}$$

and $\operatorname{WF}(u) \subset \mathbb{R}^n$ *are closed sets and under projection onto the first variable*

$$\pi_1(\operatorname{WF}(u)) = \operatorname{singsupp}(u) \subset \mathbb{R}^n,\ \pi_1(\operatorname{WF}_{\mathrm{sc}}(u)) = \operatorname{Css}(u) \subset \mathbb{B}^n. \tag{12.43}$$

Proof. To prove the first part of (12.43) we need to show that if $(\bar{x}, \omega) \notin \operatorname{WF}(u)$ for all $\omega \in \mathbb{S}^{n-1}$ with $\bar{x} \in \mathbb{R}^n$ fixed, then $\bar{x} \notin \operatorname{singsupp}(u)$. The definition (12.37) means that for each $\omega \in \mathbb{S}^{n-1}$ there exists $\phi_\omega \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ with $\phi_\omega(\bar{x}) \neq 0$ such that $\omega \notin \operatorname{Css}(\widehat{\phi_\omega u})$. Since $\operatorname{Css}(\phi u)$ is closed and $\mathbb{S}^{n-1}$ is compact, a finite number of these cutoffs, $\phi_j \in \mathcal{C}^\infty_c(\mathbb{R}^n)$, can be chosen so that $\phi_j(\bar{x}) \neq 0$ with the $\mathbb{S}^{n-1} \setminus \operatorname{Css}(\widehat{\phi_j u})$ covering $\mathbb{S}^{n-1}$. Now applying Lemma 12.13 above, we can find one

$\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, with support in $\bigcap_j \{\phi_j(x) \neq 0\}$ and $\phi(\bar{x}) \neq 0$, such that $\operatorname{Css}(\widehat{\phi u}) \subset \operatorname{Css}(\widehat{\phi_j u})$ for each j and hence $\phi u \in \mathcal{S}(\mathbb{R}^n)$ (since it is already smooth). Thus indeed it follows that $\bar{x} \notin \operatorname{singsupp}(u)$. The converse, that $\bar{x} \notin \operatorname{singsupp}(u)$ implies $(\bar{x}, \omega) \notin \operatorname{WF}(u)$ for all $\omega \in \mathbb{S}^{n-1}$ is immediate.

The argument to prove the second part of (12.43) is similar. Since, by definition, $\operatorname{WF}_{\mathrm{sc}}(u) \cap (\mathbb{R}^n \times \mathbb{B}^n) = \operatorname{WF}(u)$ and $\operatorname{Css}(u) \cap \mathbb{R}^n = \operatorname{singsupp}(u)$ we only need consider points in $\operatorname{Css}(u) \cap \mathbb{S}^{n-1}$. Now, we first check that if $\theta \notin \operatorname{Css}(u)$ then $\{\theta\} \times \mathbb{B}^n \cap \operatorname{WF}_{\mathrm{sc}}(u) = \emptyset$. By definition of $\operatorname{Css}(u)$ there is a cut-off ψ_R, where $\psi \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$ and $\psi(\theta) \neq 0$, such that $\psi_R u \in \mathcal{S}(\mathbb{R}^n)$. From (12.38) this implies that $(\theta, p) \notin \operatorname{WF}_{\mathrm{sc}}(u)$ for all $p \in \mathbb{B}^n$.

Now, Lemma 12.13 allows us to apply the same argument as used above for WF. Namely we are given that $(\theta, p) \notin \operatorname{WF}_{\mathrm{sc}}(u)$ for all $p \in \mathbb{B}^n$. Thus, for each p we may find ψ_R, depending on p, such that $\psi(\theta) \neq 0$ and $p \notin \operatorname{Css}(\widehat{\psi_R u})$. Since $\mathbb{B}^n$ is compact, we may choose a finite subset of these conic localizers, $\psi^{(j)}_{R_j}$ such that the intersection of the corresponding sets $\operatorname{Css}(\widehat{\psi^{(j)}_{R_j} u})$, is empty, i.e. their complements cover $\mathbb{B}^n$. Now, using Lemma 12.13 we may choose one ψ with support in the intersection of the sets $\{\psi^{(j)} \neq 0\}$ with $\psi(\theta) \neq 0$ and one R such that $\operatorname{Css}(\widehat{\psi_R u}) = \emptyset$, but this just means that $\psi_R u \in \mathcal{S}(\mathbb{R}^n)$ and so $\theta \notin \operatorname{Css}(u)$ as desired.

The fact that these sets are closed (in the appropriate sets) follows directly from Lemma12.13. □

Corollary 12.15. *For $u \in \mathcal{S}'(\mathbb{R}^n)$,*

$$\operatorname{WF}_{\mathrm{sc}}(u) = \emptyset \Longleftrightarrow u \in \mathcal{S}(\mathbb{R}^n). \tag{12.44}$$

Let me return to the definition of $\operatorname{WF}_{\mathrm{sc}}(u)$ and rewrite it, using what we have learned so far, in terms of a decomposition of u.

Proposition 12.16. *For any $u \in \mathcal{S}'(\mathbb{R}^n)$ and $(p, q) \in \partial(\mathbb{B}^n \times \mathbb{B}^n)$,*

$$\begin{aligned}(p,q) \notin \operatorname{WF}_{\mathrm{sc}}(u) \Longleftrightarrow \\ u = u_1 + u_2,\ u_1,\ u_2 \in \mathcal{S}'(\mathbb{R}^n),\ p \notin \operatorname{Css}(u_1),\ q \notin \operatorname{Css}(\widehat{u_2}).\end{aligned} \tag{12.45}$$

Proof. For given $(p, q) \notin \operatorname{WF}_{\mathrm{sc}}(u)$, take $\Phi = \phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ with $\phi \equiv 1$ near p, if $p \in \mathbb{R}^n$ or $\Phi = \psi_R$ with $\psi \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$ and $\psi \equiv 1$ near p, if $p \in \mathbb{S}^{n-1}$. In either case $p \notin \operatorname{Css}(u_1)$ if $u_1 = (1 - \Phi)u$ directly from the definition. So $u_2 = u - u_1 = \Phi u$. If the support of Φ is small enough it follows as in the discussion in the proof of Proposition 12.14 that

$$q \notin \operatorname{Css}(\widehat{u_2}). \tag{12.46}$$

Thus we have (12.45) in the forward direction.

For reverse implication it follows directly that $(p,q) \notin \mathrm{WF}_{\mathrm{sc}}(u_1)$ and that $(p,q) \notin \mathrm{WF}_{\mathrm{sc}}(u_2)$. □

This restatement of the definition makes it clear that there a high degree of symmetry under the Fourier transform

Corollary 12.17. *For any $u \in \mathcal{S}'(\mathbb{R}^n)$,*

$$(p,q) \in \mathrm{WF}_{\mathrm{sc}}(u)) \Longleftrightarrow (q,-p) \in \mathrm{WF}_{\mathrm{sc}}(\hat{u}). \tag{12.47}$$

Proof. I suppose a corollary should not need a proof, but still .... The statement (12.47) is equivalent to

$$(p,q) \notin \mathrm{WF}_{\mathrm{sc}}(u)) \Longrightarrow (q,-p) \notin \mathrm{WF}_{\mathrm{sc}}(\hat{u}) \tag{12.48}$$

since the reverse is the same by Fourier inversion. By (12.45) the condition on the left is equivalent to $u = u_1 + u_2$ with $p \notin \mathrm{Css}(u_1)$, $q \notin \mathrm{Css}(\widehat{u_2})$. Hence equivalent to

$$\widehat{u} = v_1 + v_2,\ v_1 = \widehat{u_2},\ \widehat{v_2} = (2\pi)^{-n}\check{u}_1 \tag{12.49}$$

so $q \notin \mathrm{Css}(v_1)$, $-p \notin \mathrm{Css}(\widehat{v_2})$ which proves (12.47). □

Now, we can exploit these notions to refine our conditions under which pairing, the product and convolution can be defined.

Theorem 12.18. *For u, $v \in \mathcal{S}'(\mathbb{R}^n)$*

(12.50) *$uv \in \mathcal{S}'(\mathbb{R}^n)$ is unambiguously defined provided*

$$(p,\omega) \in \mathrm{WF}_{\mathrm{sc}}(u) \cap (\mathbb{B}^n \times \mathbb{S}^{n-1}) \Longrightarrow (p,-\omega) \notin \mathrm{WF}_{\mathrm{sc}}(v)$$

and

(12.51) *$u * v \in \mathcal{S}'(\mathbb{R}^n)$ is unambiguously defined provided*

$$(\theta,q) \in \mathrm{WF}_{\mathrm{sc}}(u) \cap (\mathbb{S}^{n-1} \times \mathbb{B}^n) \Longrightarrow (-\theta,q) \notin \mathrm{WF}_{\mathrm{sc}}(v).$$

Proof. Let us consider convolution first. The hypothesis, (12.51) means that for each $\theta \in \mathbb{S}^{n-1}$

$$\{q \in \mathbb{B}^{n-1}; (\theta,q) \in \mathrm{WF}_{\mathrm{sc}}(u)\} \cap \{q \in \mathbb{B}^{n-1}; (-\theta,q) \in \mathrm{WF}_{\mathrm{sc}}(v)\} = \emptyset. \tag{12.52}$$

Now, the fact that $\mathrm{WF}_{\mathrm{sc}}$ is always a closed set means that (12.52) remains true near θ in the sense that if $U \subset \mathbb{S}^{n-1}$ is a sufficiently small neighbourhood of θ then

$$\{q \in \mathbb{B}^{n-1}; \exists\, \theta' \in U,\ (\theta',q) \in \mathrm{WF}_{\mathrm{sc}}(u)\} \cap \{q \in \mathbb{B}^{n-1}; \exists\, \theta'' \in U,\ (-\theta'',q) \in \mathrm{WF}_{\mathrm{sc}}(v)\} = \emptyset. \tag{12.53}$$

The compactness of $\mathbb{S}^{n-1}$ means that there is a finite cover of $\mathbb{S}^{n-1}$ by such sets U_j. Now select a partition of unity ψ_i of $\mathbb{S}^{n-1}$ which is not only subordinate to this open cover, so each ψ_i is supported in one of the U_j but satisfies the additional condition that

$$\operatorname{supp}(\psi_i) \cap (-\operatorname{supp}(\psi_{i'})) \neq \emptyset \Longrightarrow \\ \operatorname{supp}(\psi_i) \cup (-\operatorname{supp}(\psi_{i'})) \subset U_j \text{ for some } j. \tag{12.54}$$

Now, if we set $u_i = (\psi_i)_R u$, and $v_{i'} = (\psi_{i'})_R v$, we know that $u - \sum_i u_i$ has compact support and similarly for v. Since convolution is already known to be possible if (at least) one factor has compact support, it suffices to define $u_i * v_{i'}$ for every i, i'. So, first suppose that $\operatorname{supp}(\psi_i) \cap (-\operatorname{supp}(\psi_{i'})) \neq \emptyset$. In this case we conclude from (12.54) that

$$\operatorname{Css}(\widehat{u_i}) \cap \operatorname{Css}(\widehat{v_{i'}}) = \emptyset. \tag{12.55}$$

Thus we may *define*

$$\widehat{u_i * v_{i'}} = \widehat{u_i}\widehat{v_{i'}} \tag{12.56}$$

using (12.20). On the other hand if $\operatorname{supp}\psi_i \cap (-\operatorname{supp}(\psi_{i'})) = \emptyset$ then

$$\operatorname{Css}(u_i) \cap (-\operatorname{Css}(v_{i'})) \cap \mathbb{S}^{n-1} = \emptyset \tag{12.57}$$

and in this case we can define $u_i * v_{i'}$ using Lemma 12.10.

Thus with such a decomposition of u and v all terms in the convolution are well-defined. Of course we should check that this definition is independent of choices made in the decomposition. I leave this to you.

That the product is well-defined under condition (12.50) now follows if we define it using convolution, i.e. as

$$\widehat{uv} = f * g,\ \ f = \widehat{u},\ \ \check{g} = \widehat{v}. \tag{12.58}$$

Indeed, using (12.47), (12.50) for u and v becomes (12.51) for f and g. □