11. Differential operators.

In the last third of the course we will apply what we have learned about distributions, and a little more, to understand properties of differential operators with constant coefficients. Before I start talking about these, I want to prove another density result.

So far we have *not* defined a topology on $\mathcal{S}'(\mathbb{R}^n)$ – I will leave this as an optional exercise.[18] However we shall consider a notion of convergence. Suppose $u_j \in \mathcal{S}'(\mathbb{R}^n)$ is a sequence in $\mathcal{S}'(\mathbb{R}^n)$. It is said to *converge weakly* to $u \in \mathcal{S}'(\mathbb{R}^n)$ if

$$u_j(\varphi) \to u(\varphi) \;\forall\; \varphi \in \mathcal{S}(\mathbb{R}^n)\,. \tag{11.1}$$

There is no 'uniformity' assumed here, it is rather like pointwise convergence (except the linearity of the functions makes it seem stronger).

Proposition 11.1. *The subspace $\mathcal{S}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ is weakly dense, i.e., each $u \in \mathcal{S}'(\mathbb{R}^n)$ is the weak limit of a subspace $u_j \in \mathcal{S}(\mathbb{R}^n)$.*

Proof. We can use Schwartz representation theorem to write, for some m depending on u,

$$u = \langle x\rangle^m \sum_{|\alpha|\le m} D^\alpha u_\alpha\,,\ u_\alpha \in L^2(\mathbb{R}^n)\,.$$

We know that $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, in the sense of metric spaces so we can find $u_{\alpha,j} \in \mathcal{S}(\mathbb{R}^n)$, $u_{\alpha,j} \to u_\alpha$ in $L^2(\mathbb{R}^n)$. The density result then follows from the basic properties of weak convergence. □

Proposition 11.2. *If $u_j \to u$ and $u'_j \to u'$ weakly in $\mathcal{S}'(\mathbb{R}^n)$ then $cu_j \to cu$, $u_j + u'_j \to u + u'$, $D^\alpha u_j \to D^\alpha u$ and $\langle x\rangle^m u_j \to \langle x\rangle^m u$ weakly in $\mathcal{S}'(\mathbb{R}^n)$.*

Proof. This follows by writing everyting in terms of pairings, for example if $\varphi \in \mathcal{S}(\mathbb{R}^n)$

$$D^\alpha u_j(\varphi) = u_j((-1)^{(\alpha)} D^\alpha \varphi) \to u((-1)^{(\alpha)} D^\alpha \varphi) = D^\alpha u(\varphi)\,.$$

□

This weak density shows that our definition of D_j, and $x_j\times$ are unique if we require Proposition 11.2 to hold.

We have discussed differentiation as an operator (meaning just a linear map between spaces of function-like objects)

$$D_j : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)\,.$$

[18] Problem 34.

Any polynomial on $\mathbb{R}^n$

$$p(\xi) = \sum_{|\alpha|\le m} p_\alpha \xi^\alpha\,,\ p_\alpha \in \mathbb{C}$$

defines a differential operator[19]

$$p(D)u = \sum_{|\alpha|\le m} p_\alpha D^\alpha u\,. \tag{11.2}$$

Before discussing any general theorems let me consider some examples.

$$\text{On } \mathbb{R}^2,\ \overline{\partial} = \partial_x + i\partial_y \text{“d-bar operator”} \tag{11.3}$$

$$\text{on } \mathbb{R}^n,\ \Delta = \sum_{j=1}^n D_j^2 \text{“Laplacian”} \tag{11.4}$$

$$\text{on } \mathbb{R}\times\mathbb{R}^n = \mathbb{R}^{n+1},\ D_t^2 - \Delta \text{“Wave operator”} \tag{11.5}$$

$$\text{on}\mathbb{R}\times\mathbb{R}^{\mathrm{n}} = \mathbb{R}^{\mathrm{n}+1},\ \partial_{\mathrm{t}} + \Delta \text{“Heat operator”} \tag{11.6}$$

$$\text{on } \mathbb{R}\times\mathbb{R}^n = \mathbb{R}^{n+1},\ D_t + \Delta \text{“Schrödinger operator”} \tag{11.7}$$

Functions, or distributions, satisfying $\overline{\partial}u = 0$ are said to be *holomorphic*, those satisfying $\Delta u = 0$ are said to be *harmonic*.

Definition 11.3. *An element $E \in \mathcal{S}'(\mathbb{R}^n)$ satisfying*

$$P(D)E = \delta \tag{11.8}$$

is said *to be a (tempered) fundamental solution of $P(D)$.*

Theorem 11.4 (without proof)**.** *Every non-zero constant coefficient differential operator has a tempered fundamental solution.*

This is quite hard to prove and not as interetsing as it might seem. We will however give lots of examples, starting with $\overline{\partial}$. Consider the function

$$E(x,y) = \frac{1}{2\pi}(x+iy)^{-1}\,,\ (x,y) \ne 0\,. \tag{11.9}$$

Lemma 11.5. *$E(x,y)$ is locally integrable and so defines $E \in \mathcal{S}'(\mathbb{R}^2)$ by*

$$E(\varphi) = \frac{1}{2\pi}\int_{\mathbb{R}^2}(x+iy)^{-1}\varphi(x,y)\,dx\,dy\,, \tag{11.10}$$

and E so defined is a tempered fundamental solution of $\overline{\partial}$.

[19] More correctly a partial differential operator with constant coefficients.

Proof. Since $(x+iy)^{-1}$ is smooth and bounded away from the origin the local integrability follows from the estimate, using polar coordinates,

$$\int_{|(x,y)|\le 1} \frac{dx\,dy}{|x+iy|} = \int_0^{2\pi}\int_0^1 \frac{r\,dr\,d\theta}{r} = 2\pi\,. \tag{11.11}$$

Differentiating directly in the region where it is smooth,

$$\partial_x(x+iy)^{-1} = -(x+iy)^{-2}\,,\ \partial_y(x+iy)^{-1} = -i(x\in iy)^{-2}$$

so indeed, $\overline{\partial}E = 0$ in $(x,y) \neq 0$.[20]

The derivative is *really* defined by

$$\begin{aligned}(\overline{\partial}E)(\varphi) &= E(-\overline{\partial}\varphi) \\ &= \lim_{\epsilon\downarrow 0} -\frac{1}{2\pi}\int_{\substack{|x|\ge\epsilon\\|y|\ge\epsilon}} (x+iy)^{-1}\,\overline{\partial}\varphi\,dx\,dy\,.\end{aligned} \tag{11.12}$$

Here I have cut the space $\{|x|\le\epsilon\,,\ |y|\le\epsilon\}$ out of the integral and used the local integrability in taking the limit as $\epsilon\downarrow 0$. Integrating by parts in x we find

$$\begin{aligned}-\int_{\substack{|x|\ge\epsilon\\|y|\ge\epsilon}} (x+iy)^{-1}\partial_x\varphi\,dx\,dy &= \int_{\substack{|x|\ge\epsilon\\|y|\ge\epsilon}} (\partial_x(x+iy)^{-1})\varphi\,dx\,dy \\ &+ \int_{\substack{|y|\le\epsilon\\x=\epsilon}} (x+iy)^{-1}\varphi(x,y)\,dy - \int_{\substack{|y|\le\epsilon\\x=-\epsilon}} (x+iy)^{-1}\varphi(x,y)\,dy\,.\end{aligned}$$

There is a corrsponding formula for integration by parts in y so, recalling that $\overline{\partial}E = 0$ away from $(0,0)$,

$$\begin{aligned}2\pi\overline{\partial}E(\varphi) = \\ \lim_{\epsilon\downarrow 0}\int_{|y|\le\epsilon} &[(\epsilon+iy)^{-1}\varphi(\epsilon,y) - (-\epsilon+iy)^{-1}\varphi(-\epsilon,y)]\,dy \\ &+ i\lim_{\epsilon\downarrow 0}\int_{|x|\le\epsilon}[(x+i\epsilon)^{-1}\varphi(x,\epsilon) - (x-i\epsilon)^{-1}\varphi(x,\epsilon)]\,dx\,,\end{aligned} \tag{11.13}$$

assuming that both limits exist. Now, we can write

$$\varphi(x,y) = \varphi(0,0) + x\psi_1(x_1y) + y\psi_2(x,y)\,.$$

Replacing φ by either $x\psi_1$ or $y\psi_2$ in (11.13) both limits are zero. For example

$$\left|\int_{|y|\le\epsilon} (\epsilon+iy)^{-1}\epsilon\psi_1(\epsilon,y)\,dy\right| \le \int_{|y|\le\epsilon}|\psi_1| \to 0\,.$$

[20] Thus at this stage we know $\overline{\partial}E$ must be a sum of derivatives of δ.

Thus we get the same result in (11.13) by replacing $\varphi(x,y)$ by $\varphi(0,0)$. Then $2\pi\overline{\partial}E(\varphi) = c\varphi(0)$,

$$c = \lim_{\epsilon\downarrow 0} 2\epsilon \int_{|y|\le\epsilon} \frac{dy}{\epsilon^2+y^2} = \lim_{\epsilon\downarrow 0} < \int_{|y|\le 1} \frac{dy}{1+y^2} = 2\pi\,.$$

□

Let me remind you that we have already discussed the convolution of functions

$$u * v(x) = \int u(x-y)v(y)\,dy = v * u(x)\,.$$

This makes sense provided u is of slow growth and $s \in \mathcal{S}(\mathbb{R}^n)$. In fact we can rewrite the definition in terms of pairing

$$(u*\varphi)(x) = \langle u, \varphi(x-\cdot)\rangle \tag{11.14}$$

where the $\cdot$ indicates the variable in the pairing.

Theorem 11.6 (Hörmander, Theorem 4.1.1)**.** *If $u \in \mathcal{S}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ then $u*\varphi \in \mathcal{S}'(\mathbb{R}^n)\cap\mathcal{C}^\infty(\mathbb{R}^n)$ and if* $\operatorname{supp}(\varphi) \Subset \mathbb{R}^n$

$$\operatorname{supp}(u*\varphi) \subset \operatorname{supp}(u) + \operatorname{supp}(\varphi)\,.$$

For any multi-index α

$$D^\alpha(u*\varphi) = D^\alpha u * \varphi = u * D^\alpha\varphi\,.$$

Proof. If $\varphi \in \mathcal{S}(\mathbb{R}^n)$ then for any fixed $x \in \mathbb{R}^n$,

$$\varphi(x-\cdot) \in \mathcal{S}(\mathbb{R}^n)\,.$$

Indeed the seminorm estimates required are

$$\sup_y (1+|y|^2)^{k/2}\,|D^\alpha{}_y\varphi(x-y)| < \infty \ \forall\ \alpha, k > 0\,.$$

Since $D^\alpha{}_y\varphi(x-y) = (-1)^{|\alpha|}(D^\alpha\varphi)(x-y)$ and

$$(1+|y|^2) \le (1+|x-y|^2)(1+|x|^2)$$

we conclude that

$$\|(1+|y|^2)^{k/2}D^\alpha{}_y(x-y)\|_{L^\infty} \le (1+|x|^2)^{k/2}\|\langle y\rangle^k D^\alpha{}_y\varphi(y)\|_{L^\infty}\,.$$

The continuity of $u \in \mathcal{S}'(\mathbb{R}^n)$ means that for some k

$$|u(\varphi)| \le C \sup_{|\alpha|\le k} \|(y)^k D^\alpha\varphi\|_{L^\infty}$$

so it follows that

$$|u*\varphi(x)| = |\langle u, \varphi(x-\cdot)\rangle| \le C(1+|x|^2)^{k/2}\,. \tag{11.15}$$

The argument above shows that $x \mapsto \varphi(x - \cdot)$ is a continuous function of $x \in \mathbb{R}^n$ with values in $\mathcal{S}(\mathbb{R}^n)$, so $u * \varphi$ is continuous and satisfies (11.15). It is therefore an element of $\mathcal{S}'(\mathbb{R}^n)$.

Differentiability follows in the same way since for each j, with e_j the jth unit vector

$$\frac{\varphi(x + se_j - y) - \varphi(x - y)}{s} \in \mathcal{S}(\mathbb{R}^n)$$

is continuous in $x \in \mathbb{R}^n$, $s \in \mathbb{R}$. Thus, $u * \varphi$ has continuous partial derivatives and

$$D_j u * \varphi = u * D_j \varphi\,.$$

The same argument then shows that $u * \varphi \in \mathcal{C}^\infty(\mathbb{R}^n)$. That $D_j(u * \varphi) = D_j u * \varphi$ follows from the definition of derivative of distributions

$$\begin{aligned} D_j(u * \varphi(x)) &= (u * D_j\varphi)(x) \\ &= \langle u, D_{x_j}\varphi(x - y)\rangle = -\langle u(y), D_{y_j}\varphi(x - y)\rangle_y \\ &= (D_j u) * \varphi\,. \end{aligned}$$

Finally consider the support property. Here we are assuming that $\operatorname{supp}(\varphi)$ is compact; we also know that $\operatorname{supp}(u)$ is a closed set. We have to show that

$$\overline{x} \notin \operatorname{supp}(u) + \operatorname{supp}(\varphi) \tag{11.16}$$

implies $u * \varphi(x') = 0$ for x' near $\overline{x}$. Now (11.16) just means that

$$\operatorname{supp}\varphi(\overline{x} - \cdot) \cap \operatorname{supp}(u) = \phi\,, \tag{11.17}$$

Since $\operatorname{supp}\varphi(x - \cdot) = \{y \in \mathbb{R}^n; x - y \in \operatorname{supp}(\varphi)\}$, so both statements mean that there is *no* $y \in \operatorname{supp}(\varphi)$ with $\overline{x} - y \in \operatorname{supp}(u)$. This can also be written

$$\operatorname{supp}(\varphi) \cap \operatorname{supp} u(x - \cdot) = \phi$$

and as we showed when discussing supports implies

$$u * \varphi(x') = \langle u(x' - \cdot), \varphi\rangle = 0\,.$$

From (11.17) this is an *open* condition on x', so the support property follows.

□

Now suppose φ, $\psi \in \mathcal{S}(\mathbb{R}^n)$ and $u \in \mathcal{S}'(\mathbb{R}^n)$. Then

$$(u * \varphi) * \psi = u * (\varphi * \psi)\,. \tag{11.18}$$

This is really Hörmander's Lemma 4.1.3 and Theorem 4.1.2; I ask you to prove it as Problem 35.

We have shown that $u * \varphi$ is $\mathcal{C}^\infty$ if $u \in \mathcal{S}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$, i.e., the regularity of $u * \varphi$ follows from the regularity of *one* of the factors. This makes it reasonable to expect that $u * v$ can be defined when $u \in \mathcal{S}'(\mathbb{R}^n)$, $v \in \mathcal{S}'(\mathbb{R}^n)$ and one of them has compact support. If $v \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ then

$$u * v(\varphi) = \int \langle u(\cdot), v(x - \cdot)\rangle \varphi(x)\, dx = \int \langle u(\cdot), v(x - \cdot)\rangle \check{v}\varphi(-x)\, dx$$

where $\check{\varphi}(z) = \varphi(-z)$. In fact using Problem 35,

$$u * v(\varphi) = ((u * v) * \check{\varphi})(0) = (u * (v * \check{\varphi}))(0)\,. \tag{11.19}$$

Here, v, φ are both smooth, but notice

Lemma 11.7. *If $v \in \mathcal{S}'(\mathbb{R}^n)$ has compact support and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ then $v * \varphi \in \mathcal{S}(\mathbb{R}^n)$.*

Proof. Since $v \in \mathcal{S}'(\mathbb{R}^n)$ has compact support there exists $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ such that $\chi v = v$. Then

$$\begin{aligned} v * \varphi(x) = (\chi v) * \varphi(x) &= \langle \chi v(y), \varphi(x - y)\rangle_y \\ &= \langle u(y), \chi(y)\varphi(x - y)\rangle_y\,. \end{aligned}$$

Thus, for some k,

$$|v * \varphi(x)| \le C\|\chi(y)\varphi(x - y)\|_{(k)}$$

where $\|\ \|_{(k)}$ is one of our norms on $\mathcal{S}(\mathbb{R}^n)$. Since χ is supported in some large ball,

$$\begin{aligned} &\|\chi(y)\varphi(x - y)\|_{(k)} \\ &\quad \le \sup_{|\alpha| \le k} \left|\langle y\rangle^k D^\alpha{}_y(\chi(y)\varphi(x - y))\right| \\ &\quad \le C \sup_{|y| \le R} \sup_{|\alpha| \le k} |(D^\alpha \varphi)(x - y)| \\ &\quad \le C_N \sup_{|y| \le R} (1 + |x - y|^2)^{-N/2} \\ &\quad \le C_N (1 + |x|^2)^{-N/2}\,. \end{aligned}$$

Thus $(1 + |x|^2)^{N/2}\, |v * \varphi|$ is bounded for each N. The same argument applies to the derivative using Theorem 11.6, so

$$v * \varphi \in \mathcal{S}(\mathbb{R}^n)\,.$$

□

In fact we get a little more, since we see that for each k there exists k' and C (depending on k and v) such that

$$\|v * \varphi\|_{(k)} \le C\|\varphi\|_{(k')}\,.$$

This means that

$$v* : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$$

is a continuous linear map.

Now (11.19) allows us to define $u*v$ when $u \in \mathcal{S}'(\mathbb{R}^n)$ and $v \in \mathcal{S}'(\mathbb{R}^n)$ has compact support by

$$u * v(\varphi) = u * (v * \check{\varphi})(0)\,.$$

Using the continuity above, I ask you to check that $u * v \in \mathcal{S}'(\mathbb{R}^n)$ in Problem 36. For the moment let me assume that this convolution has the same properties as before – I ask you to check the main parts of this in Problem 37.

Recall that $E \in \mathcal{S}'(\mathbb{R}^n)$ is a fundamental situation for $P(D)$, a constant coefficient differential operator, if $P(D)E = \delta$. We also use a weaker notion.

Definition 11.8. *A* parametrix *for a constant coefficient differential operator $P(D)$ is a distribution $F \in \mathcal{S}'(\mathbb{R}^n)$ such that*

$$P(D)F = \delta + \psi\,,\ \psi \in \mathcal{C}^\infty(\mathbb{R}^n)\,. \tag{11.20}$$

An operator $P(D)$ is said to be hypoelliptic *if it has a parametrix satisfying*

$$\operatorname{sing\,supp}(F) \subset \{0\}\ , \tag{11.21}$$

where for any $u \in \mathcal{S}'(\mathbb{R}^n)$

$$(\operatorname{sing\,supp}(u))^{\complement} = \{\overline{x} \in \mathbb{R}^n;\ \exists\, \varphi \in \mathcal{C}^\infty_c(\mathbb{R}^n)\,,\ \varphi(\overline{x}) \neq 0, \varphi u \in \mathcal{C}^\infty_c(\mathbb{R}^n)\}\ . \tag{11.22}$$

Since the same φ must work for nearby points in (11.22), the set $\operatorname{sing\,supp}(u)$ is *closed*. Furthermore

$$\operatorname{sing\,supp}(u) \subset \operatorname{supp}(u)\,. \tag{11.23}$$

As Problem 37 I ask you to show that if $K \Subset \mathbb{R}^n$ and $K \cap \operatorname{sing\,supp}(u) = \phi$ the $\exists\ \varphi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ with $\varphi(x) = 1$ in a neighbourhood of K such that $\varphi u \in \mathcal{C}^\infty_c(\mathbb{R}^n)$. In particular

$$\operatorname{sing\,supp}(u) = \phi \Rightarrow u \in \mathcal{S}'(\mathbb{R}^n) \cap \mathcal{C}^\infty(\mathbb{R}^n)\,. \tag{11.24}$$

Theorem 11.9. *If $P(D)$ is hypoelliptic then*

$$\operatorname{sing\,supp}(u) = \operatorname{sing\,supp}(P(D)u)\ \forall\ u \in \mathcal{S}'(\mathbb{R}^n)\,. \tag{11.25}$$

Proof. One half of this is true for *any* differential operator:

Lemma 11.10. *If $u \in \mathcal{S}'(\mathbb{R}^n)$ then for any polynomial*

$$\operatorname{sing\,supp}(P(D)u) \subset \operatorname{sing\,supp}(u)\ \forall\ u \in \mathcal{S}'(\mathbb{R}^n)\,. \tag{11.26}$$

□

Proof. We must show that $\overline{x} \notin \operatorname{sing\,supp}(u) \Rightarrow \overline{x} \notin \operatorname{sing\,supp}(P(D)u)$. Now, if $\overline{x} \notin \operatorname{sing\,supp}(u)$ we can find $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, $\varphi \equiv 1$ near $\overline{x}$, such that $\varphi u \in \mathcal{C}_c^\infty(\mathbb{R}^n)$. Then

$$\begin{aligned} P(D)u &= P(D)(\varphi u + (1-\varphi)u) \\ &= P(D)(\varphi u) + P(D)((1-\varphi)u)\,. \end{aligned}$$

The first term is $\mathcal{C}^\infty$ and $\overline{x} \notin \operatorname{supp}(P(D)((1-\varphi)u))$, so $\overline{x} \notin \operatorname{sing\,supp}(P(D)u)$. □

It remains to show the converse of (11.26) where $P(D)$ is assumed to be hypoelliptic. Take F, a parametrix for $P(D)$ with $\operatorname{sing\,supp} u \subset \{0\}$ and assume, or rather arrange, that F have compact support. In fact if $\overline{x} \notin \operatorname{sing\,supp}(P(D)u)$ we can arrange that

$$(\operatorname{supp}(F) + \overline{x}) \cap \operatorname{sing\,supp}(P(D)u) = \phi\,.$$

Now $P(D)F = \delta\psi$ with $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ so

$$u = \delta * u = (P(D)F) * u - \psi * u.$$

Since $\psi * u \in \mathcal{C}^\infty$ it suffices to show that $\bar{x} \notin \operatorname{sing\,supp}\left((P(D)u) * f\right)$.

Take $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ with $\varphi f \in \mathcal{C}^\infty$, $f = P(D)u$ but

$$(\operatorname{supp} F + \overline{x}) \cap \operatorname{supp}(\varphi) = 0\,.$$

Then $f = f_1 + f_2$, $f_1 = \varphi f \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ so

$$f * F = f_1 * F + f_2 * F$$

where $f_1 * F \in \mathcal{C}^\infty(\mathbb{R}^n)$ and $\overline{x} \notin \operatorname{supp}(f_2 * F)$. It follows that $\overline{x} \notin \operatorname{sing\,supp}(u)$.

Example 11.1. If u is holomorphic on $\mathbb{R}^n$, $\overline{\partial}u = 0$, then $u \in \mathcal{C}^\infty(\mathbb{R}^n)$.

Recall from last time that a differential operator $P(D)$ is said to be hypoelliptic if there exists $F \in \mathcal{S}'(\mathbb{R}^n)$ with

(11.27) $$P(D)F - \delta \in \mathcal{C}^\infty(\mathbb{R}^n) \text{ and } \operatorname{sing\,supp}(F) \subset \{0\}\ .$$

The second condition here means that if $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ and $\varphi(x) = 1$ in $|x| < \epsilon$ for some $\epsilon > 0$ then $(1-\varphi)F \in \mathcal{C}^\infty(\mathbb{R}^n)$. Since $P(D)((1-\varphi)F) \in \mathcal{C}^\infty(\mathbb{R}^n)$ we conclude that

$$P(D)(\varphi F) - \delta \in \mathcal{C}_c^\infty(\mathbb{R}^n)$$

and we may well suppose that F, replaced now by φF, has compact support. Last time I showed that

$$\text{If } P(D) \text{ is hypoelliptic and } u \in \mathcal{S}'(\mathbb{R}^n) \text{ then}$$
$$\operatorname{sing\,supp}(u) = \operatorname{sing\,supp}(P(D)u)\,.$$

I will remind you of the proof later.

First however I want to discuss the important notion of *ellipticity*. Remember that $P(D)$ is 'really' just a polynomial, called the *characteristic polynomial*

$$P(\xi) = \sum_{|\alpha|\le m} C_\alpha \xi^\alpha .$$

It has the property

$$\widehat{P(D)u}(\xi) = P(\xi)\hat{u}(\xi) \ \forall\ u \in \mathcal{S}'(\mathbb{R}^n) .$$

This shows (if it isn't already obvious) that we can remove $P(\xi)$ from $P(D)$ thought of as an operator on $\mathcal{S}'(\mathbb{R}^n)$.

We can think of *inverting* $P(D)$ by dividing by $P(\xi)$. This works well provided $P(\xi) \neq 0$, for all $\xi \in \mathbb{R}^n$. An example of this is

$$P(\xi) = |\xi|^2 + 1 = \sum_{j=1}^n +1 .$$

However even the Laplacian, $\Delta = \sum_{j=1}^n D_j^2$, does not satisfy this rather stringent condition.

It is reasonable to expect the top order derivatives to be the most important. We therefore consider

$$P_m(\xi) = \sum_{|\alpha|=m} C_\alpha \xi^\alpha$$

the leading part, or *principal symbol*, of $P(D)$.

Definition 11.11. *A polynomial $P(\xi)$, or $P(D)$, is said to be* elliptic *of order m provided $P_m(\xi) \neq 0$ for all $0 \neq \xi \in \mathbb{R}^n$.*

So what I want to show today is

Theorem 11.12. *Every elliptic differential operator $P(D)$ is hypoelliptic.*

We want to find a *parametrix* for $P(D)$; we already know that we might as well suppose that F has compact support. Taking the Fourier transform of (11.27) we see that $\widehat{F}$ should satisfy

$$P(\xi)\widehat{F}(\xi) = 1 + \widehat{\psi},\ \widehat{\psi} \in \mathcal{S}(\mathbb{R}^n) . \tag{11.28}$$

Here we use the fact that $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$, so $\widehat{\psi} \in \mathcal{S}(\mathbb{R}^n)$ too.

First suppose that $P(\xi) = P_m(\xi)$ is actually homogeneous of degree m. Thus

$$P_m(\xi) = |\xi|^m P_m(\widehat{\xi}),\ \widehat{\xi} = \xi / |\xi| \ , \xi \neq 0 .$$

The assumption at ellipticity means that

$$P_m(\widehat{\xi}) \neq 0 \;\forall\; \widehat{\xi} \in \mathcal{S}^{n-1} = \{\xi \in \mathbb{R}^n; |\xi| = 1\} \;. \tag{11.29}$$

Since $\mathcal{S}^{n-1}$ is *compact* and P_m is continuous

$$\left|P_m(\widehat{\xi})\right| \geq C > 0 \;\forall\; \widehat{\xi} \in \mathcal{S}^{n-1}\,, \tag{11.30}$$

for some constant C. Using homogeneity

$$\left|P_m(\widehat{\xi})\right| \geq C\left|\xi\right|^m \,,\; C > 0 \;\forall\; \xi \in \mathbb{R}^n\,. \tag{11.31}$$

Now, to get $\widehat{F}$ from (11.28) we want to divide by $P_m(\xi)$ or multiply by $1/P_m(\xi)$. The only problem with defining $1/P_m(\xi)$ is at $\xi = 0$. We shall simply avoid this unfortunate point by choosing $P \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ as before, with $\varphi(\xi) = 1$ in $|\xi| \leq 1$.

Lemma 11.13. *If $P_m(\xi)$ is homogeneous of degree m and elliptic then*

$$Q(\xi) = \frac{(1 - \varphi(\xi))}{P_m(\xi)} \in \mathcal{S}'(\mathbb{R}^n) \tag{11.32}$$

is the Fourier transform of a parametrix for $P_m(D)$, satisfying (11.27).

Proof. Clearly $Q(\xi)$ is a continuous function and $|Q(\xi)| \leq C(1+|\xi|)^{-m} \;\forall\; \xi \in \mathbb{R}^n$, so $Q \in \mathcal{S}'(\mathbb{R}^n)$. It therefore *is* the Fourier transform of some $F \in \mathcal{S}'(\mathbb{R}^n)$. Furthermore

$$\begin{aligned} \widehat{P_m(D)}F(\xi) &= P_m(\xi)\widehat{F} = P_m(\xi)Q(\xi) \\ &= 1 - \varphi(\xi)\,, \\ \Rightarrow P_m(D)F &= \delta + \psi\,,\; \widehat{\psi}(\xi) = -\varphi(\xi)\,. \end{aligned}$$

Since $\varphi \in \mathcal{C}^\infty_c(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$, $\psi \in \mathcal{S}(\mathbb{R}^n) \subset \mathcal{C}^\infty(\mathbb{R}^n)$. Thus F *is* a parametrix for $P_m(D)$. We still need to show the 'hard part' that

$$\operatorname{sing\,supp}(F) \subset \{0\} \;. \tag{11.33}$$

□

We can show (11.33) by considering the distributions $x^\alpha F$. The idea is that for $|\alpha|$ large, x^α vanishes rather rapidly at the origin and this should 'weaken' the singularity of F there. In fact we shall show that

$$x^\alpha F \in H^{|\alpha|+m-n-1}(\mathbb{R}^n)\,,\; |\alpha| > n + 1 - m\,. \tag{11.34}$$

If you recall, these Sobolev spaces are defined in terms of the Fourier transform, namely we must show that

$$\widehat{x^\alpha F} \in \langle\xi\rangle^{-|\alpha|-m+n+1}L^2(\mathbb{R}^n)\,.$$

Now $\widehat{x^\alpha F} = (-1)^{|\alpha|} D^\alpha{}_\xi \widehat{F}$, so what we need to cinsider is the behaviour of the derivatives of $\widehat{F}$, which is just $Q(\xi)$ in (11.32).

Lemma 11.14. *Let $P(\xi)$ be a polynomial of degree m satisfying*

$$|P(\xi)| \geq C\,|\xi|^m \ \ in\ \ |\xi| > 1/C \ for\ some\ C > 0\,, \tag{11.35}$$

then for some constants C_α

$$\left| D^\alpha \frac{1}{P(\xi)} \right| \leq C_\alpha\, |\xi|^{-m-|\alpha|} \ \ in\ \ |\xi| > 1/C\,. \tag{11.36}$$

Proof. The estimate in (11.36) for $\alpha = 0$ is just (11.35). To prove the higher estimates that for each α there is a polynomial of degree at most $(m-1)\,|\alpha|$ such that

$$D^\alpha \frac{1}{P(\xi)} = \frac{L_\alpha(\xi)}{(P(\xi))^{1+|\alpha|}}\,. \tag{11.37}$$

Once we know (11.37) we get (11.36) straight away since

$$\left| D^\alpha \frac{1}{P(\xi)} \right| \leq \frac{C'_\alpha\,|\xi|^{(m-1)|\alpha|}}{C^{1+|\alpha|}\,|\xi|^{m(1+|\alpha|)}} \leq C_\alpha\,|\xi|^{-m-|\alpha|}\ .$$

We can prove (11.37) by induction, since it is certainly true for $\alpha = 0$. Suppose it is true for $|\alpha| \leq k$. To get the same identity for each β with $|\beta| = k+1$ it is enough to differentiate one of the identities with $|\alpha| = k$ once. Thus

$$D^\beta \frac{1}{P(\xi)} = D_j D^\alpha \frac{1}{P(\xi)} = \frac{D_j L_\alpha(\xi)}{P(\xi)^{1+|\alpha|}} - \frac{(1+|\alpha|) L_\alpha D_j P(\xi)}{(P(\xi))^{2+|\alpha|}}\,.$$

Since $L_\beta(\xi) = P(\xi) D_j L_\alpha(\xi) - (1+|\alpha|) L_\alpha(\xi) D_j P(\xi)$ is a polynomial of degree at most $(m-1)\,|\alpha| + m - 1 = (m-1)\,|\beta|$ this proves the lemma. □

Going backwards, observe that $Q(\xi) = \frac{1-\varphi}{P_m(\xi)}$ is smooth in $|\xi| \leq 1/C$, so (11.36) implies that

$$|D^\alpha Q(\xi)| \leq C_\alpha (1+|\xi|)^{-m-|\alpha|} \tag{11.38}$$

$$\Rightarrow \langle \xi \rangle^\ell D^\alpha Q \in L^2(\mathbb{R}^n) \text{ if } \ell - m - |\alpha| < -\frac{n}{2}\,,$$

which certainly holds if $\ell = |\alpha| + m - n - 1$, giving (11.34). Now, by Sobolev's embedding theorem

$$x^\alpha F \in \mathcal{C}^k \text{ if } |\alpha| > n + 1 - m + k + \frac{n}{2}\,.$$

In particular this means that if we choose $\mu \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ with $0 \notin \operatorname{supp}(\mu)$ then for every k, $\mu/|x|^{2k}$ is smooth and

$$\mu F = \frac{\mu}{|x|^{2k}} |x|^{2k} F \in \mathcal{C}^{2\ell-2n}\,,\ \ell > n\,.$$

Thus $\mu F \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ and this is what we wanted to show, $\operatorname{sing\,supp}(F) \subset \{0\}$.

So now we have actually proved that $P_m(D)$ is hypoelliptic if it is elliptic. Rather than go through the proof again to make sure, let me go on to the general case and in doing so review it.

Proof. Proof of theorem. We need to show that if $P(\xi)$ is elliptic then $P(D)$ has a parametrix F as in (11.27). From the discussion above the ellipticity of $P(\xi)$ implies (and is equivalent to)

$$|P_m(\xi)| \ge c\,|\xi|^m\,,\ c > 0\,.$$

On the other hand

$$P(\xi) - P_m(\xi) = \sum_{|\alpha|<m} C_\alpha \xi^\alpha$$

is a polynomial of degree at most $m-1$, so

$$|P(\xi) - P_m(\xi)|\,2 \le C'(1+|\xi|)^{m-1}\,.$$

This means that id $C > 0$ is large enough then in $|\xi| > C$, $C'(1+|\xi|)^{m-1} < \frac{c}{2}|\xi|^m$, so

$$\begin{aligned}|P(\xi)| &\ge |P_m(\xi)| - |P(\xi) - P_m(\xi)|\\ &\ge c\,|\xi|^m - C'(1+|\xi|)^{m-1} \ge \frac{c}{2}|\xi|^m\,.\end{aligned}$$

This means that $P(\xi)$ itself satisfies the conditions of Lemma 11.14. Thus if $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ is equal to 1 in a large enough ball then $Q(xi) = (1-\varphi(\xi))/P(\xi)$ in $\mathcal{C}^\infty$ and satisfies (11.36) which can be written

$$|D^\alpha Q(\xi)| \le C_\alpha (1+|\xi|)^{m-|\alpha|}\,.$$

The discussion above now shows that defining $F \in \mathcal{S}'(\mathbb{R}^n)$ by $\widehat{F}(\xi) = Q(\xi)$ gives a solution to (11.27).

□

The last step in the proof is to show that if $F \in \mathcal{S}'(\mathbb{R}^n)$ has compact support, and satisfies (11.27), then

$$\begin{aligned}&u \in \mathcal{S}(\mathbb{R}^n)\,,\ P(D)u \in \mathcal{S}'(\mathbb{R}^n) \cap \mathcal{C}^\infty(\mathbb{R}^n)\\ &\Rightarrow u = F * (P(D)u) - \psi * u \in \mathcal{C}^\infty(\mathbb{R}^n)\,.\end{aligned}$$

Let me refine this result a little bit.

Proposition 11.15. *If $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\mu \in \mathcal{S}'(\mathbb{R}^n)$ has compact support then*

$$\operatorname{sing\,supp}(u * f) \subset \operatorname{sing\,supp}(u) + \operatorname{sing\,supp}(f).$$

Proof. We need to show that $p \notin \operatorname{sing\,supp}(u) \in \operatorname{sing\,supp}(f)$ then $p \notin \operatorname{sing\,supp}(u * f)$. Once we can fix p, we might as well suppose that f has compact support too. Indeed, choose a large ball $B(R, 0)$ so that

$$z \notin B(0, R) \Rightarrow p \notin \operatorname{supp}(u) + B(0, R)\,.$$

This is possible by the assumed boundedness of $\operatorname{supp}(u)$. Then choose $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ with $\varphi = 1$ on $B(0, R)$; it follows from Theorem L16.2, or rather its extension to distributions, that $\phi \notin \operatorname{supp}(u(1-\varphi)f)$, so we can replace f by φf, noting that $\operatorname{sing\,supp}(\varphi f) \subset \operatorname{sing\,supp}(f)$. Now if f has compact support we can choose compact neighbourhoods K_1, K_2 of $\operatorname{sing\,supp}(u)$ and $\operatorname{sing\,supp}(f)$ such that $p \notin K_1 + K_2$. Furthermore we an decompose $u = u_1 + u_2$, $f = f_1 + f_2$ so that $\operatorname{supp}(u_1) \subset K_1$, $\operatorname{supp}(f_2) \subset K_2$ and $u_2, f_2 \in \mathcal{C}^\infty(\mathbb{R}^n)$. It follows that

$$u * f = u_1 * f_1 + u_2 * f_2 + u_1 * f_2 + u_2 * f_2\,.$$

Now, $p \notin \operatorname{supp}(u_1 * f_1)$, by the support property of convolution and the three other terms are $\mathcal{C}^\infty$, since at least one of the factors is $\mathcal{C}^\infty$. Thus $p \notin \operatorname{sing\,supp}(u * f)$. □

The most important example of a differential operator which is hypoelliptic, but not elliptic, is the heat operator

$$\partial_t + \Delta = \partial_t - \sum_{j=1}^n \partial_{x_j}^2\,. \tag{11.39}$$

In fact the distribution

$$E(t, x) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x|^2}{4t}\right) & t \geq 0 \\ 0 & t \leq 0 \end{cases} \tag{11.40}$$

is a fundamental solution. First we need to check that E *is* a distribution. Certainly E is $\mathcal{C}^\infty$ in $t > 0$. Moreover as $t \downarrow 0$ in $x \neq 0$ it vanishes with all derivatives, so it is $\mathcal{C}^\infty$ except at $t = 0$, $x = 0$. Since it is clearly measurable we will check that it is locally integrable near the origin, i.e.,

$$\int_{\substack{0 \leq t \leq 1 \\ |x| \leq 1}} E(t, x)\, dx\, dt < \infty\,, \tag{11.41}$$

since $E \geq 0$. We can change variables, setting $X = x/t^{1/2}$, so $dx = t^{n/2}\,dX$ and the integral becomes

$$\frac{1}{(4\pi)^{n/2}}\int_0^{t'}\int_{|X|\leq t^{-1/2}} \exp(-\frac{|X|^2}{4})\,dx\,dt \ < \infty\,.$$

Since E is actually bounded near infinity, it follows that $E \in \mathcal{S}'\mathbb{R}^n$,

$$E(\varphi) = \int_{t\geq 0} E(t,x)\varphi(t,x)\,dx\,dt \ \forall\ \varphi \in \mathcal{S}(\mathbb{R}^{n+1})\,.$$

As before we want to compute

$$(\partial_t + \Delta)E(\varphi) = E(-\partial_t\varphi + \Delta\varphi) \tag{11.42}$$
$$= \lim_{\mathcal{E}\downarrow 0}\int_{\mathcal{E}}^{\infty}\int_{\mathbb{R}^n} E(t,x)(-\partial_t\varphi + \Delta\varphi)\,dx\,dt\,.$$

First we check that $(\partial_t + \Delta)E = 0$ in $t > 0$, where it is a $\mathcal{C}^\infty$ function. This is a straightforward computation:

$$\partial_t E = -\frac{n}{2t}E + \frac{|x|^2}{4t^2}E$$
$$\partial_{x_j}E = -\frac{x_j}{2t}E\,,\ \partial^2_{x_j}E = -\frac{1}{2t}E + \frac{x_j^2}{4t^2}E$$
$$\Rightarrow \Delta E = \frac{n}{2t}E + \frac{|x|^2}{4t^2}E\,.$$

Now we can integrate by parts in (11.42) to get

$$(\partial_t + \Delta)E(\varphi) = \lim_{\mathcal{E}\downarrow 0}\int_{\mathbb{R}^n}\varphi(\mathcal{E},x)\frac{e^{-|x|^2/4\mathcal{E}}}{(4\pi\mathcal{E})^{n/2}}\,dx\,.$$

Making the same change of variables as before, $X = x/2\mathcal{E}^{1/2}$,

$$(\partial_t + \Delta)E(\varphi) = \lim_{\mathcal{E}\downarrow 0}\int_{\mathbb{R}^n}\varphi(\mathcal{E},\mathcal{E}^{1/2}X)\frac{e^{-|x|^2}}{\pi^{n/2}}\,dX\,.$$

As $\mathcal{E} \downarrow 0$ the integral here is bounded by the integrable function $C\exp(-|X|^2)$, for some $C > 0$, so by Lebesgue's theorem of dominated convergence, conveys to the integral of the limit. This is

$$\varphi(0,0)\cdot\int_{\mathbb{R}^n} e^{-|x|^2}\frac{dx}{\pi^{n/2}} = \varphi(0,0)\,.$$

Thus

$$(\partial_t + \Delta)E(\varphi) = \varphi(0,0) \Rightarrow (\partial_t + \Delta)E = \delta_t\delta_x\,,$$

so E is indeed a fundamental solution. Since it vanishes in $t < 0$ it is canned a *forward fundamrntal* solution.

Let's see what we can use it for.

Proposition 11.16. *If* $f \in \mathcal{S}'\mathbb{R}^n$ *has compact support* $\exists\,!u \in \mathcal{S}'\mathbb{R}^n$ *with* $\operatorname{supp}(m) \subset \{t \geq -T\}$ *for some* T *and*

$$(\partial_t + \Delta)u = f \ \text{in}\ \mathbb{R}^{n+1}\,. \tag{11.43}$$

Proof. Naturally we try $u = E * f$. That it satisfies (11.43)follows from the properties of convolution. Similarly if T is such that $\operatorname{supp}(f) \subset \{t \geq T\}$ then

$$\operatorname{supp}(u) \subset \operatorname{supp}(f) + \operatorname{supp}(E) \subset \{t \geq T]\ .$$

So we need to show *uniqueness*. If $u_1, u_2 \in \mathcal{S}'\mathbb{R}^n$ in two solutions of (11.43) then their difference $v = u_1 - u_2$ satisfies the 'homogeneous' equation $(\partial_t + \Delta)v = 0$. Furthermore, $v = 0$ in $t < T'$ for some T'. Given any $E \in \mathbb{R}$ choose $\varphi(t) \in \mathcal{C}^\infty(\mathbb{R})$ with $\varphi(t) = 0$ in $t > \bar{t} + 1$, $\varphi(t) = 1$ in $t < \bar{t}$ and consider

$$E_{\bar{t}} = \varphi(t)E = F_1 + F_2\,,$$

where $F_1 = \psi E_{\bar{t}}$ for some $\psi \in \mathcal{C}_c^\infty\mathbb{R}^{n+1})$, $\psi = 1$ near 0. Thus F_1 has comapct support and in fact $F_2 \in \mathcal{S}\mathbb{R}^n$. I ask you to check this last statement as Problem L18.P1.

Anyway,

$$(\partial_t + \Delta)(F_1 + F_2) = \delta + \psi \in \mathcal{S}\mathbb{R}^n\,,\ \psi_{\bar{t}} = 0\ t \leq \bar{t}\,.$$

Now,

$$(\partial_t + \Delta)(E_t * u) = 0 = u + \psi_{\bar{t}} * u\,.$$

Since $\operatorname{supp}(\psi_{\bar{t}}) \subset \left\{t \geq \bar{t}\right]$, the second tier here is supported in $t \geq \bar{t} \geq T'$. Thus $u = 0$ in $t < \bar{t} + T'$, but $\bar{t}$ is arbitrary, so $u = 0$. □

Notice that the assumption that $u \in \mathcal{S}'\mathbb{R}^n$ is not redundant in the statement of the Proposition, if we allow "large" solutions they become non-unique. Problem L18.P2 asks you to apply the fundamental solution to solve the initial value problem for the heat operator.

Next we make similar use of the fundamental solution for Laplace's operator. If $n \geq 3$ the

$$E = C_n\,|x|^{-n+2} \tag{11.44}$$

is a fundamental solution. You should check that $\Delta E_n = 0$ in $x \neq 0$ directly, I will show later that $\Delta E_n = \delta$, for the appropriate choice of C_n, but you can do it directly, as in the case $n = 3$.

Theorem 11.17. *If* $f \in \mathcal{S}\mathbb{R}^n$ $\exists\ !u \in \mathcal{C}_0^\infty\mathbb{R}^n$ *such that* $\Delta u = f$.

Proof. Since convolution $u = E * f \in \mathcal{S}'\mathbb{R}^n \cap \mathcal{C}^\infty\mathbb{R}^n$ is defined we certainly get a solution to $\Delta u = f$ this way. We need to check that $u \in \mathcal{C}_0^\infty\mathbb{R}^n$. First we know that Δ is hypoelliptic so we can decompose

$$E = F_1 + F_2\,,\ F_1 \in \mathcal{S}'\mathbb{R}^n\,, \operatorname{supp} F, \Subset \mathbb{R}^n$$

and then $F_2 \in \mathcal{C}^\infty\mathbb{R}^n$. In fact we can see from (11.44) that

$$|D^\alpha F_2(x)| \le C_\alpha(1+|x|)^{-n+2-|\alpha|}\,.$$

Now, $F_1 * f \in \mathcal{S}\mathbb{R}^n$, as we showed before, and continuing the integral we see that

$$\begin{aligned}|D^\alpha u| &\le |D^\alpha F_2 * f| + C_N(1+|x|)^{-N}\ \forall\ N\\ &\le C'_\alpha(1+|x|)^{-n+2-|\alpha|}\,.\end{aligned}$$

Since $n > 2$ it follows that $u \in \mathcal{C}_0^\infty\mathbb{R}^n$.

So only the uniqueness remains. If there are two solutions, u_1, u_2 for a given f then $v = u_1 - u_2 \in \mathcal{C}_0^\infty\mathbb{R}^n$ satisfies $\Delta v = 0$. Since $v \in \mathcal{S}'\mathbb{R}^n$ we can take the Fourier transform and see that

$$|\chi|^2\,\widehat{v}(\chi) = 0 \Rightarrow \operatorname{supp}(\widehat{v}) \subset \{0\}\ .$$

an earlier problem was to conclude from this that $\widehat{v} = \sum_{|\alpha|\le m} C_\alpha D^\alpha\delta$ for some constants C_α. This in turn implies that v is a polynomial. However the only polynomials in $\mathcal{C}_0^0\mathbb{R}^n$ are identically 0. Thus $v = 0$ and uniqueness follows. □