Problem set 7

Due November 16, 2004. Problems from notes:- 61, 62, plus the following two problems also at the end of the problems in the notes as numbers 75, 76.

Restriction from Sobolev spaces. The Sobolev embedding theorem shows that a function in $H^m(\mathbb{R}^n),$ for $m > n/2$ is continuous - and hence can be restricted to a subspace of $\mathbb{R}^n.$ In fact this works more generally. Show that there is a well defined *restriction map*

$$H^m(\mathbb{R}^n) \longrightarrow H^{m-\frac{1}{2}}(\mathbb{R}^n) \text{ if } m > \frac{1}{2} \tag{8}$$

with the following properties:

1. On $\mathcal{S}(\mathbb{R}^n)$ it is given by $u \longmapsto u(0, x'),\ x' \in \mathbb{R}^{n-1}.$
2. It is continuous and linear.

Hint: Use the usual method of finding a weak version of the map on smooth Schwartz functions; namely show that in terms of the Fourier transforms on $\mathbb{R}^n$ and $\mathbb{R}^{n-1}$

$$\widehat{u(0,\cdot)}(\xi') = (2\pi)^{-1} \int_{\mathbb{R}} \hat{u}(\xi_1, \xi')d\xi_1,\ \forall\ \xi' \in \mathbb{R}^{n-1}. \tag{9}$$

Use Cauchy's inequality to show that this is continuous as a map on Sobolev spaces as indicated and then the density of $\mathcal{S}(\mathbb{R}^n)$ in $H^m(\mathbb{R}^n)$ to conclude that the map is well-defined and unique.

Restriction by WF: From class we know that the product of two distributions, one with compact support, is defined provided they have no `opposite' directions in their wavefront set:

$$(x,\omega) \in \mathrm{WF}(u) \Longrightarrow (x,-\omega) \notin \mathrm{WF}(v) \text{ then } uv \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n). \tag{10}$$

Show that this product has the property that $f(uv) = (fu)v = u(fv)$ if $f \in \mathcal{C}^\infty(\mathbb{R}^n)$.
Use this to define a restriction map to $x_1 = 0$ for distributions of compact support satisfying $((0, x'), (\omega_1, 0)) \notin \mathrm{WF}(u)$ as the product

$$u_0 = u\delta(x_1). \tag{11}$$

[Show that $u_0(f), f \in \mathcal{C}^\infty(\mathbb{R}^n)$ only depends on $f(0, \cdot) \in \mathcal{C}^\infty(\mathbb{R}^{n-1})$.