Problem set 3: Due October 5

From notes: Problems 16, 17, 18, 21, 24

Problem 1 Let $\| \ \|$ be a norm on a vector space V. Show that $\|u\| = (u,u)^{1/2}$ for an inner product satisfying the conditions of a pre-Hilbert space if and only if the parallelogram law holds for every pair $u, v \in V.$

Hint (From Dimitri Kountourogiannis)

If $\| \cdot \|$ comes from an inner product, then it must satisfy the polarisation identity:

$$(x,y) = 1/4(\|x+y\|^2 - \|x-y\|^2 - i\|x+iy\|^2 - i\|x-iy\|^2)$$

i.e, the inner product is recoverable from the norm, so use the RHS (right hand side) to define an inner product on the vector space. You will need the paralellogram law to verify the additivity of the RHS. Note the polarization identity is a bit more transparent for real vector spaces. There we have

$$(x,y) = 1/2(\|x+y\|^2 - \|x-y\|^2)$$

both are easy to prove using $\|a\|^2 = (a,a).$

Problem 2 Show (Rudin does it) that if $u : \mathbb{R}^n \to \mathbb{C}$ has continuous partial derivatives then it is differentiable at each point.

Problem 3 Consider the function $f(x) = \langle x \rangle^{-1} = (1+|x|^2)^{-1/2}$. Show that

$$\frac{\partial f}{\partial x_j} = l_j(x) \cdot \langle x \rangle^{-3}$$

with $l_j(x)$ a linear function. Conclude by *induction* that $\langle x \rangle^{-1} \in \mathcal{C}_0^k(\mathbb{R}^n)$ for *all* k.

Problem 4 Show that a linear *map*

$$T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$$

is continuous if and only if for *each* k there exist C and j such that if $|\alpha| \le k$ and $|\beta| \le k$

$$\sup \left| x^{\alpha} D^{\beta} T\varphi \right| \leq C \sum_{|\alpha'| \leq j,\ |\beta'| \leq j} \sup_{\mathbb{R}^n} \left| x^{\alpha'} D^{\beta'} \varphi \right| \ \forall\ \varphi \in \mathcal{S}(\mathbb{R}^n). \tag{3}$$

Problem 5 Show that elements of $L^2(\mathbb{R}^n)$ are ``continuous in the mean'' i.e.,

$$\lim_{|t| \to 0} \int_{\mathbb{R}^n} |u(x+t) - u(x)|^2 \ dx = 0\,. \tag{4}$$