Problem set 2: Due September 28

From Notes: Problems 6, 11, 12, 13, 14.

Problem 1 Show that the smallest σ-algebra containing the sets

$$(a, \infty] \subset [-\infty, \infty]$$

for all $a \in \mathbb{R}$, is what is called above the `Borel' σ-algebra on $[-\infty, \infty]$.

Problem 2 Let $(X, \mathcal{M}, \mu)$ be any measure space (so μ is a measure on the σ-algebra $\mathcal{M}$ of subsets of X). Show that the set of equivalence classes of μ-integrable functions on X, with the equivalence relation

$$f_1 \equiv f_2 \Longleftrightarrow \mu(\{x \in X; f_1(x) \neq f_2(x)\}) = 0.$$

is a normed linear space with the usual linear structure and the norm given by

$$\|f\| = \int_X |f| d\mu.$$

Problem 3 Let $(X, \mathcal{M})$ be a set with a σ-algebra. Let $\mu : \mathcal{M} \to \mathbb{R}$ be a finite measure in the sense that $\mu(\phi) = 0$ and for any $\{E_i\}_{i=1}^{\infty} \subset \mathcal{M}$ with $E_i \cap E_j = \phi$ for $i \neq j$,

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i) \tag{1}$$

with the series on the right *always* absolutely convergenct (i.e., this is part of the requirement on μ). Define

$$|\mu|(E) = \sup \sum_{i=1}^{\infty} |\mu(E_i)| \tag{2}$$

for $E \in \mathcal{M}$, with the supremum over *all* measurable decompositions $E = \bigcup_{i=1}^{\infty} E_i$ with the E_i disjoint. Show that $|\mu|$ is a finite, positive measure.

Hint 1. You must show that $|\mu|(E)=\sum_{i=1}^{\infty}|\mu|(A_i)$ if $\bigcup_i A_i = E$, $A_i \in \mathcal{M}$ being disjoint. Observe that if $A_j = \bigcup_l A_{jl}$ is a measurable decomposition of A_j then together the A_{jl} give a decomposition of E. Similarly, if $E = \bigcup_j E_j$ is any such decomposition of E then $A_{jl} = A_j \cap E_l$ gives such a decomposition of A_j.

Hint 2. See W. Rudin, *Real and complex analysis*, third edition ed., McGraw-Hill, 1987. p. 117!

Problem 4 (Hahn Decomposition)

With assumptions as in Problem 3:

1. Show that $\mu_+ = \frac{1}{2}(|\mu| + \mu)$ and $\mu_- = \frac{1}{2}(|\mu| - \mu)$ are positive measures, $\mu = \mu_+ - \mu_-$. Conclude that the definition of a measure in the notes based on (4.17) is the *same* as that in Problem 3.
2. Show that $\mu_\pm$ so constructed are orthogonal in the sense that there is a set $E \in \mathcal{M}$ such that $\mu_-(E) = 0, \mu_+(X \setminus E) = 0.$

   Hint. Use the definition of $|\mu|$ to show that for any $F \in \mathcal{M}$ and any $\epsilon > 0$ there is a subset $F' \in \mathcal{M}, F' \subset F$ such that $\mu_+(F') \geq \mu_+(F) - \epsilon$ and $\mu_-(F') \leq \epsilon.$ Given $\delta > 0$ apply this result repeatedly (say with $\epsilon = 2^{-n}\delta)$ to find a decreasing sequence of sets $F_1 = X, F_n \in \mathcal{M}, F_{n+1} \subset F_n$ such that $\mu_+(F_n) \geq \mu_+(F_{n-1}) - 2^{-n}\delta$ and $\mu_-(F_n) \leq 2^{-n}\delta.$ Conclude that $G = \bigcap_n F_n$ has $\mu_+(G) \geq \mu_+(X) - \delta$ and $\mu_-(G) = 0.$ Now let G_m be chosen this way with $\delta = 1/m.$ Show that $E = \bigcup_m G_m$ is as required.

Problem 5

Now suppose that μ is a finite, positive Radon measure on a locally compact metric space X (meaning a finite positive Borel measure outer regular on Borel sets and inner regular on open sets). Show that μ is inner regular on all Borel sets and hence, given $\epsilon > 0$ and

$E \in \mathcal{B}(X)$ there exist sets $K \subset E \subset U$ with K compact and U open such that $\mu(K) \geq \mu(E) - \epsilon$, $\mu(E) \geq \mu(U) - \epsilon$.

Hint. First take U open, then use *its* inner regularity to find K with $K' \Subset U$ and $\mu(K') \geq \mu(U) - \epsilon/2$. How big is $\mu(E \backslash K')$? Find $V \supset K' \backslash E$ with V open and look at $K = K' \backslash V$.