Problem set 1

Due September 23, 1PM. (Sorry about saying September 16!) Problems 1-5, 10 (I am dropping 11 because I will not get far enough) from the notes. Here they are, with the wording changed a bit.

Problem 1 (Notes 1) Prove that u_+, defined by (1.10) is linear.
Problem 2 (Notes 2) Prove Lemma I.8.

Hint(s). All functions here are supposed to be continuous, I just don't bother to keep on saying it.

1. Recall, or check, that the local compactness of a metric space X means that for each point $x \in X$ there is an $\epsilon > 0$ such that the ball $\{y \in X; d(x,y) \leq \delta\}$ is compact for $\delta \leq \epsilon.$
2. First do the case $n = 1,$ so $K \Subset U$ is a compact set in an open subset.
    1. Given $\delta > 0,$ use the local compactness of $X,$ to cover K with a finite number of compact closed balls of radius at most $\delta.$
    2. Deduce that if $\epsilon > 0$ is small enough then the set $\{x \in X; d(x,K) \leq \epsilon\},$ where

       $$d(x,K) = \inf_{y \in K} d(x,y),$$

       is compact.
    3. Show that $d(x,K),$ for K compact, is continuous.
    4. Given $\epsilon > 0$ show that there is a continuous function $g_\epsilon : \mathbb{R} \longrightarrow [0,1]$ such that $g_\epsilon(t) = 1$ for $t \leq \epsilon/2$ and $g_\epsilon(t) = 0$ for $t > 3\epsilon/4.$
    5. Show that $f = g_\epsilon \circ d(\cdot, K)$ satisfies the conditions for $n = 1$ if $\epsilon > 0$ is small enough.
3. Prove the general case by induction over $n.$
    1. In the general case, set $K' = K \cap U_1^{\complement}$ and show that the inductive hypothesis applies to K' and the U_j for $j > 1;$ let $f'_j, j = 2, \ldots, n$ be the functions supplied by the inductive assumption and put $f' = \sum_{j \geq 2} f'_j.$

2. Show that $K_1 = K \cap \{f' \le \frac{1}{2}\}$ is a compact subset of U_1.
3. Using the case $n = 1$ construct a function F for K_1 and U_1.
4. Use the case $n = 1$ again to find G such that $G = 1$ on K and $\mathrm{supp}(G) \Subset \{f' + F > \frac{1}{2}\}$.
5. Make sense of the functions

$$f_1 = F\frac{G}{f' + F},\ f_j = f_j'\frac{G}{f' + F},\ j \ge 2$$

and show that they satisfies the inductive assumptions.

Problem 3 (Notes 3) (Easy) Show that σ-algebras are closed under countable intersections.

Problem 4 (Notes 4) (Easy) Show that if μ is a complete measure and $E \subset F$ where F is measurable and has measure 0 then $\mu(E) = 0$.

Problem 5 (Notes 5) Show that (in a locally compact metric space) compact subsets are measurable for any Borel measure. (This just means that compact sets are Borel sets if you follow through the tortuous terminology.)

Problem 6 (Notes 10) For the space $Y = \mathbb{N} = \{1, 2, \dots\} \subset \mathbb{R}$, describe $C_0(Y)$ and guess a description of its dual in terms of sequences.