MEASURE AND INTEGRATION: LECTURE 9

Invariance of Lebesgue measure. Given $A \subset \mathbb{R}^n$ and $z \in \mathbb{R}^n$, let $z + A = \{z + x \mid x \in A\}$ be the *translate* of A by z. Given $t > 0$, let $tA = \{tx \mid x \in A\}$ be the *dilation* of A by t.

Let $I = [a_1, b_1] \times \cdots \times [a_n, b_n]$ and $z = z_1 \times \cdots \times z_n$. Then

$$z + I = [z_1 + a_1, z_1 + b_1] \times \cdots \times [z_n + a_n, z_n + b_n],$$

and

$$tI = [ta_1, tb_1] \times \cdots \times [ta_n, tb_n],$$

and we have

$$\begin{aligned} \lambda(z + I) &= (z_1 + b_1 - z_1 - a_1) \cdots (z_n + b_n - z_n - a_z) \\ &= (b_1 - a_1) \cdots (b_n - a_n) \\ &= \lambda(I). \end{aligned}$$

and

$$\lambda(tI) = t^n \cdot \lambda(I).$$

If P is a special polygon, then $\lambda(z + P) = \lambda(P)$ and $\lambda(tP) = t^n P$. Indeed, write $P = \sum_{i=1}^{N} I_i$ and the proof is straightforward.

If G is an open set, then $\lambda(z + G) = \lambda(G)$ and $\lambda(tG) = t^n \lambda(G)$. We have $\lambda(G) = \sup\{\lambda(P) \mid P \subset G \text{ special polygon}\}$, so $\lambda(z + G) = \sup\{\lambda(P) \mid P \subset z + G, P \text{ special polygon}\}$. But $P \subset G$ special polygon $\iff z + P \subset z + G$ special polygon. Since Lebesgue invariance holds for special polygons, it holds for open sets.

Finally, by similar reasoning, it can be shown that a set $A \subset \mathbb{R}^n$ is measurable if and only if $z + A$ is measurable if and only if tA is measurable, and $\lambda(A) = \lambda(z + A)$, $\lambda(tA) = t^n \lambda(A)$.

A non-measurable set $E \subset \mathbb{R}^n$. Let $\mathbb{Q}$ be the set of rational numbers. For $x \in \mathbb{R}$, consider $x + \mathbb{Q} = \{x + q \mid q \in \mathbb{Q}\}$. Then $y \in x + \mathbb{Q} \iff y - x \in \mathbb{Q}$.

Claim: if $x, x' \in \mathbb{R}$, then either (i) $x + \mathbb{Q} = x' + \mathbb{Q}$ or (ii) $(x + \mathbb{Q}) \cap (x' + \mathbb{Q}) = \emptyset$. Proof: If the intersection is nonempty, then there exists $y = x + q_1 = x' + q_2$, which implies that $x - x' = q_1 - q_2 \in \mathbb{Q}$. Thus, $x + \mathbb{Q} = x' + \mathbb{Q}$, and the claim is proved.

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We have shown that $\mathbb{R}$ is covered disjointly by the sets $x + \mathbb{Q}$.

The *Axiom of Choice* states that there exists a set $E \subset \mathbb{R}$ such that every point of $\mathbb{R}$ belongs to only one of these sets, i.e.,

$$\mathbb{R} = \bigcup_{x \in E} (x + \mathbb{Q})$$

is a disjoint union. Alternatively, for any $x \in \mathbb{R}$, there exists a unique $y \in E$ and unique $z \in \mathbb{Q}$ such that $x = y + z$.

Since the set $\mathbb{Q}$ is countable, its elements can be enumerated: $\mathbb{Q} = \{q_1, q_2, \ldots\}$. Thus,

$$\mathbb{R} = \bigcup_{k=1}^{\infty} (q_k + E)$$

is a disjoint union. Using outer measure subadditivity and invariance of Lebesgue measure,

$$\lambda^*(\mathbb{R}) \leq \sum_{k=1}^{\infty} \lambda^*(q_k + E) = \sum_{k=1}^{\infty} \lambda^*(E).$$

Hence we must have that $\lambda^*(E) > 0$ (otherwise $\lambda^*(\mathbb{R}) = 0$).

Now let $K \subset E$ be an arbitrary compact subset of E and let $D = (0, 1) \cap \mathbb{Q}$. (The set D is a bounded countably infinite set.) Then

$$\bigcup_{q \in D} (q + K) = D + K$$

is a bounded set. The sets in the union are disjoint, since rational translates of E are disjoint. We have

$$\begin{aligned}
\infty &> \lambda(D + K) \quad \text{(bounded)} \\
&= \lambda\left(\bigcup_{q \in D} (q + K)\right) \\
&= \sum_{q \in D} \lambda(q + K) \\
&= \sum_{q \in D} \lambda(K).
\end{aligned}$$

Since the sum is over an infinite index set, $\lambda(K) = 0$. Because $K \subset E$ arbitrary $\Rightarrow \lambda(K) = 0$, we have $\lambda_*(E) = 0$. But $0 = \lambda_(E) < \lambda^*(E) \Rightarrow E \notin \mathcal{L}$.

Corollary 0.1. *If $A \subset \mathbb{R}^n$ is measurable with positive measure, then there exists $B \subset A$ that is not measurable.*

Proof. Write $A = \cup_{k=1}^{\infty} ((q_k + E) \cap A)$ as a disjoint union. Then

$$0 < \lambda(A) = \lambda^*(A) \leq \sum_{k=1}^{\infty} \lambda^* ((q_k + E) \cap A),$$

and so $\lambda^* ((q_k + E) \cap A) > 0$ for some k. But $\lambda_* ((q_k + E) \cap A) \leq \lambda_*(q_k + E) = \lambda_*(E) = 0$, a contradiction. □

Invariance under linear transformations.

Theorem 0.2. *Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear map and $A \subset \mathbb{R}^n$. Then*

$$\lambda^*(TA) = |\det T| \, \lambda^*(A),$$
$$\lambda_*(TA) = |\det T| \, \lambda_*(A).$$

If A is measurable, then TA is measurable and

$$\lambda(TA) = |\det T| \, \lambda(A).$$

Proof. First assume that T is invertible, i.e., that $\det T \neq 0$. We will use the following lemma.

Lemma 0.3. *Let T be invertible and let $J = [0, 1)^n$. Let ρ be defined by $\lambda(TJ) = \rho\lambda(J)$. If $A \subset \mathbb{R}^n$, then $\lambda^*(TA) = \rho\lambda^*(A)$ and $\lambda_*(TA) = \rho\lambda_*(A)$. If A is measurable, then TA is measurable and $\lambda(TA) = \rho\lambda(A)$.*

Proof. The set J is the union of countably many compact sets:

$$J = \bigcup_{k=1}^{\infty} [0, 1 - 1/k]^n,$$

and so

$$TJ = \bigcup_{k=1}^{\infty} T([0, 1 - 1/k]).$$

Since T maps compact sets to compact sets, TJ is the union of countably many compact sets. Thus, TJ is measurable, so the definition of ρ makes sense.

We just to need to prove that $\lambda(TG) = \rho\lambda(G)$ for G open. As before, if the measure of open sets is invariant, then outer measure, compacts, and inner measures are invariant.

Let $G \subset \mathbb{R}^n$ be open. Claim: can write $G = \cup_{k=1}^{\infty} J_k$ with J_k's disjoint and each J_k is a translation and dilation of J. (Pair by integer of those not contained, then pair by 1/2, then by 1/4, ...) Let $J_k = z_k + t_k \cdot J$. Then $\lambda(J_k) = t_k^n \lambda(J)$.

$$TJ_k = Tz_k + t_k \cdot TJ$$

$$\begin{aligned} \Rightarrow \lambda(TJ_k) &= t_k^n \lambda(TJ) \\ &= t_k^n \rho \lambda(J) \\ &= t_k^n \rho t_k^{1-n} \lambda(J_k). \end{aligned}$$

Thus, $\lambda(TJ_k) = \rho\lambda(J_k)$. Since $G = \cup_{k=1}^{\infty} J_k$, $TG = \cup_{k=1}^{\infty} TJ_k$, which is a disjoint collection of measurable sets. Thus we have

$$\lambda(TG) = \sum_{k=1}^{\infty} \lambda(TJ_k) = \sum_{k=1}^{\infty} \rho \cdot \lambda(J_k) = \rho \cdot \lambda(G).$$

□

To identify ρ, check for elementary matrices just on the cube. This shows that in fact $\rho = |\det T|$.

Lastly, if T is not invertible, i.e., $\det T = 0$, then the image $T\mathbb{R}^n$ is the subset of a hyperplane. This means that TA has measure zero, so the formula still holds. □

A linear transformation is a rotation when the matrix is an *orthogonal* matrix: $AA^T = I$. In this case, it must be that $\det A = \pm 1$. Thus, Lebesgue measure is invariant under rotation.

Finally, there is an important subgroup of the group of all $n \times n$ real matrices known as the special linear group, denoted

$$SL(n, \mathbb{R}) = \{A \mid \det A = 1\}.$$