MEASURE AND INTEGRATION: LECTURE 7

Review. The steps to defining Lebesgue measure. (1) measure of rectangles (2) measure of special polygons (3) measure of open sets: $\lambda(G) = \sup\{\lambda(P) \mid P \subset G, P \text{ special polygon}\}$. (4) measure of compact sets: $\lambda(K) = \inf\{\lambda(G) \mid K \subset G, G \text{ open}\}$. (5) Inner λ_* and outer λ^* measures.

Lebesgue measurable sets (with finite outer measure). Let $A \subset \mathbb{R}^n$ and $\lambda^*(A) < \infty$ (A has finite outer measure). Then we write that $A \in \mathcal{L}_0 \iff \lambda^*(A) = \lambda_*(A)$ and define measure of A to be

$$\lambda(A) = \lambda^*(A) = \lambda_*(A).$$

We know that $\mathcal{L}_0$ contains all compact sets and open sets of finite measure.

Lemma 0.1. *Let $A, B \in \mathcal{L}_0$. If A and B are disjoint, then $A \cup B \in \mathcal{L}_0$ and $\lambda(A \cup B) = \lambda(A) + \lambda(B)$.*

Proof.

$$\begin{aligned}
\lambda^*(A \cup B) &\leq \lambda^*(A) + \lambda^*(B) && \text{(Outer measure subadditivity)} \\
&= \lambda(A) + \lambda(B) && (A, B \in \mathcal{L}_0) \\
&= \lambda_*(A) + \lambda_*(B) && \text{(Property of inner measure)} \\
&\leq \lambda_*(A \cup B) \leq \lambda^*(A \cup B)
\end{aligned}$$

□

Main approximation theorem.

Theorem 0.2. *Let $A \in \mathbb{R}^n$ and $\lambda^*(A) < \infty$. Then $A \in \mathcal{L}_0$ if and only if for all $\epsilon > 0$ there exists K compact and G open such that $K \subset A \subset G$ and $\lambda(G \setminus K) < \epsilon$.*

Proof. If $A \in \mathcal{L}_0$, then there exists $G \supset A$ open such that $\lambda(G) < \lambda^*(A) + \epsilon/2$ and there exists $K \subset A$ compact such that $\lambda(K) > \lambda_*(A) - \epsilon/2$. Since $K \subset G$, we can write $G = K \cup (G \setminus K)$ as a disjoint union of sets in $\mathcal{L}_0$, and so $\lambda(G) = \lambda(K) + \lambda(G \setminus K)$. That is,

$$\lambda(G \setminus K) = \lambda(G) - \lambda(K) < \lambda(A) + \epsilon/2 - (\lambda(A) - \epsilon/2) = \epsilon.$$

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For the other direction, fix $\epsilon > 0$ and choose $K \subset A \subset G$ such that $\lambda(G \setminus K) < \epsilon$. Then

$$\lambda^*(A) \leq \lambda(G) = \lambda(K) + \lambda(G \setminus K) \leq \lambda(K) + \epsilon \leq \lambda_*(A) + \epsilon.$$

Since this holds for any $\epsilon > 0$, we have $\lambda^*(A) \leq \lambda_*(A) \leq \lambda^*(A)$, and hence $\lambda_*(A) = \lambda^*(A)$. □

Corollary 0.3. *If $A, B \in \mathcal{L}_0$, then $A \cup B$, $A \cap B$, and $A \setminus B$ are all in $\mathcal{L}_0$.*

Proof. By the approximation theorem, for any $\epsilon > 0$, we can find $K_1 \subset A \subset G_1$ and $K_2 \subset B \subset G_2$ such hat $\lambda(G_1 \setminus K_1) < \epsilon/2$ and $\lambda(G_2 \setminus K_2) < \epsilon/2$. Then $K_1 \cup K_2 \subset A \cup B \subset G_1 \cup G_2$, and so

$$\begin{aligned}(G_1 \cup G_2) \setminus (K_1 \cup K_2) &= (G_1 \cup G_2) \cap (K_1 \cup K_2)^c \\ &= G_1 \cap (K_1 \cup K_2)^c \cup G_2 \cap (K_1 \cup K_2)^c \\ &\subset G_1 \cap K_1^c \cup G_2 \cap K_2^c \\ &= (G_1 \setminus K_1) \cup (G_2 \setminus K_2).\end{aligned}$$

Thus,

$$\begin{aligned}\lambda(G_1 \cup G_2 \setminus (K_1 \cup K_2)) &\leq \lambda(G_1 \setminus K_1) + \lambda(G_2 \setminus K_2) \\ &< \epsilon/2 + \epsilon/2 = \epsilon,\end{aligned}$$

and $A \cup B \in \mathcal{L}_0$. Let K_i, G_i $(i = 1, 2)$ be as before. Then $K_1 \cap K_2 \subset A \cap B \subset G_1 \cap G_2$. We have

$$\begin{aligned}(G_1 \cap G_2) \setminus (K_1 \cap K_2) &= (G_1 \cap G_2) \cap (K_1 \cap K_2)^c \\ &= (G_1 \cap G_2) \cap (K_1^c \cup K_2^c) \\ &= (G_1 \cap G_2 \cap K_1^c) \cup (G_1 \cap G_2 \cap K_2^c) \\ &\subset (G_1 \cap K_1^c) \cup (G_2 \cap K_2^c) \\ &\subset (G_1 \setminus K_1) \cup (G_2 \setminus K_2).\end{aligned}$$

Thus,

$$\begin{aligned}\lambda(G_1 \cap G_2 \setminus (K_1 \cap K_2)) &< \lambda(G1 \setminus K_1) + \lambda(G_2 \setminus K_2) \\ &< \epsilon/2 + \epsilon/2 = \epsilon.\end{aligned}$$

The proof for $A \setminus B$ is similar. □

Countable additivity. Let $A_k \in \mathcal{L}_0$ for $k = 1, 2, \ldots$. Let $A = \cup_{k=1}^{\infty} A_k$ and assume $\lambda^*(A) < \infty$. Then $A \in \mathcal{L}_0$ and $\lambda(A) \leq \sum_{k=1}^{\infty} \lambda(A_k)$. Furthermore, if the A_k's are disjoint, then $\lambda(A) = \sum_{k=1}^{\infty} \lambda(A_k)$.

Proof. First, the disjoint case. We have

$$\begin{aligned}\lambda^*(A) &\le \sum_{k=1}^{\infty} \lambda^*(A_k) \quad \text{(outer measure subadditivity)}\\ &= \sum_{k=1}^{\infty} \lambda_*(A_k) \qquad \text{(each } A_k \in \mathcal{L}_0\text{)}\\ &\le \lambda_*\left(\bigcup_{k=1}^{\infty} A_k\right) \le \lambda^*(A).\end{aligned}$$

Since $\lambda_*(A) = \lambda^*(A)$, $A \in \mathcal{L}_0$, and it also follows that $\lambda(A) = \sum_{k=1}^{\infty} \lambda(A_k)$.

In general, rewrite A as a disjoint union as follows. Let $B_1 = A_1$, $B_2 = A_2 \setminus A_1$, $B_3 = A_3 \setminus (A_1 \cup A_2)$, and so on. Each $B_k \in \mathcal{L}_0$, clearly the B_k's are disjoint. It is straightforward to check that $A = \cup_{k=1}^{\infty} B_k$: the fact that the union is a subset of A is obvious, and if $x \in A_k$, then $x \in B_1$ or B_2 or $\ldots B_k$. From the preceding disjoint case we know that $\cup_{k=1}^{\infty} B_k \in \mathcal{L}_0$, and

$$\lambda(A) = \lambda\left(\bigcup_{k=1}^{\infty} B_k\right) = \sum_{k=1}^{\infty} \lambda(B_k) \le \sum_{k=1}^{\infty} \lambda(A_k),$$

where in the last step we noticed that each $B_k \subset A_k$. □

Extension to arbitrary measurable sets. Let $A \subset \mathbb{R}^n$. Then A is Lebesgue measurable (and we write $A \in \mathcal{L}$) if for all $M \in \mathcal{L}_0$, we have $A \cap M \in \mathcal{L}_0$. In this case, define

$$\lambda(A) = \sup_{M \in \mathcal{L}_0} \{\lambda(A \cap M)\}.$$

Proposition 0.4. *The new λ is consistent with all λ when $\lambda^* < \infty$. In other words, if $A \subset \mathbb{R}^n$ and $\lambda(A) < \infty$, then $A \in \mathcal{L}_0 \iff A \in \mathcal{L}$, and the definitions of $\lambda(A)$ agree.*

Proof. If $A \in \mathcal{L}_0$, then the lemma implies that $A \cap M \in \mathcal{L}_0$ for all $M \in \mathcal{L}_0$; thus $A \in \mathcal{L}$. We know $B(0,k)$, the ball of radius k with center at the origin, is in $\mathcal{L}_0$. Let $A_k = A \cap B(0,k)$. Then by definition of $\mathcal{L}$, we have $A_k \in \mathcal{L}$. Also, $A = \cup_{k=1}^{\infty} A_k \in \mathcal{L}_0$ by the countable subadditivity theorem.

Next, take $A \in \mathcal{L}_0$. Let $\tilde{\lambda}(A)$ to be the new definition, that is,

$$\tilde{\lambda}(A) = \sup_{M \in \mathcal{L}_0} \{\lambda(A \cap M)\}.$$

Then $A \cap M \subset \Rightarrow \lambda(A \cap M) \le \lambda(A) \Rightarrow \tilde{\lambda}(A) \le \lambda(A)$. Since $A \in \mathcal{L}_0$, choose $M = A$ in definition of $\tilde{\lambda}$. Then $\tilde{\lambda}(A) \ge \lambda(A)$, and thus equality must hold. □

Properties of (arbitrary) Lebesgue measurable sets.

(1) $A \in \mathcal{L} \Rightarrow A^c \in \mathcal{L}_0$
(2) If $A_i \in L$ $(i = 1, 2, \ldots)$, then $\cup_{i=1}^{\infty} A_i \in \mathcal{L}$.
(3) If A_i disjoint, then $\lambda\left(\cup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \lambda(A_i)$.

Proof. (1) For $M \in \mathcal{L}_0$, NTS that $A^c \cap M \in \mathcal{L}_0$. We know $A \cap M \in \mathcal{L}_0$. Since $A^c \cap M = M \setminus (A \cap M)$, and both $M \in \mathcal{L}_0$ and $A \cap M \in \mathcal{L}_0$, we are done.

(2) For $M \in \mathcal{L}_0$, $A_i \cap M \in \mathcal{L}_0$. We have $(\cup_{i=1}^{\infty} A_i) \cap M = \cup_{i=1}^{\infty}(A_i \cap M)$ and by countable additivity the last term is in $\mathcal{L}_0$.

(3) Let $A = \cup_{k=1}^{\infty} A_k$. Then $A \cap M = \cup_{k=1}^{\infty}(A_k \cap M)$ is a disjoint union. Thus, $\lambda(A \cap M) = \sum_{k=1}^{\infty} \lambda(A_k \cap M) \leq \sum_{k=1}^{\infty} \lambda(A_k)$. Taking sup over all M gives $\lambda(A) \leq \sum_{k=1}^{\infty} \lambda(A_k)$. For the other direction, fix N. Let $M_1, \ldots, M_N \in \mathcal{L}_0$ be arbitrary and put $M = \cup_{k=1}^{N} M_k$. Then

$$\begin{aligned}
\lambda(A) \geq \lambda(A \cap M) &= \lambda\left(\bigcup_{k=1}^{\infty}(A_k \cap M)\right) \\
&= \sum_{k=1}^{\infty} \lambda(A_k \cap M) \\
&\geq \sum_{k=1}^{N} \lambda(A_k \cap M) \\
&\geq \sum_{k=1}^{N} \lambda(A_k \cap M_k).
\end{aligned}$$

Since M_k are arbitrary, taking sup over all M_k gives $\lambda(A) \geq \sum_{k=1}^{N} \lambda(A_k)$. Letting $N \to \infty$, $\lambda(A) \geq \sum_{k=1}^{\infty} \lambda(A_k)$.

□

Corollary 0.5. *By (1) and (2), $\mathcal{L}$ is a σ-algebra.*

Corollary 0.6. *By (3), λ is a positive measure on $\mathcal{L}$, and thus $(\mathbb{R}^n, \mathcal{L}, \lambda)$ is a measure space.*