MEASURE AND INTEGRATION: LECTURE 6

Lebesgue measure on $\mathbb{R}^n$. We will define the Lebesgue measure λ: {subsets of $\mathbb{R}^n$} $\to [0,\infty]$ through a series of steps.

(1) $\lambda(\emptyset) = 0$.

(2) Special rectangles: rectangles with sides parallel to axes.
- $n = 1$: $\lambda([a,b]) = b - a$
- $n = 2$: $\lambda([a_1,b_1] \times [a_2,b_2]) = (b_1 - a_1)(b_2 - a_2)$.

$$\vdots$$

(3) Special polygons: finite unions of special rectangles. To find measure, write $P = \cup_{k=1}^N I_k$, I_k disjoint special rectangles. Define $\lambda(P) = \sum_{k=1}^N \lambda(I_k)$.

Properties of Lebesgue measure.

(1) Well-defined. If $P = \cup_{k=1}^N I_k = \cup_{k=1}^{N'} I'_k$, then $\sum_{k=1}^N \lambda(I_k) = \sum_{k=1}^{N'} \lambda(I'_k)$. (Exercise)

(2) $P_1 \subset P_2 \Rightarrow \lambda(P_1) \leq \lambda(P_2)$.

(3) P_1, P_2 disjoint $\Rightarrow \lambda(P_1 \cup P_2) = \lambda(P_1) + \lambda(P_2)$.

Open sets. Let $G \subset \mathbb{R}^n$ be open and nonempty. We will approximate G from within by special polygons. That is, we define

$$\lambda(G) = \sup\{\lambda(P) \mid P \subset G, P \text{ special polygon}\}.$$

Properties for open sets.

(1) $\lambda(G) = 0 \iff G = \emptyset$. (Nontrivial open sets have positive measure.)

(2) $\lambda(\mathbb{R}^n) = \infty$.

(3) $G_1 \subset G_2 \Rightarrow \lambda(G_1) \leq \lambda(G_2)$.

(4) $\lambda\left(\cup_{k=1}^\infty G_k\right) \leq \sum_{k=1}^\infty \lambda(G_k)$.

(5) G_k open and pairwise disjoint $\Rightarrow \lambda\left(\cup_{k=1}^\infty G_k\right) = \sum_{k=1}^\infty \lambda(G_k)$.

(6) P special polygon $\Rightarrow \lambda(P) = \lambda(P^\circ)$, where P° = interior of P.

Proof. (3) If $P \subset G_1$, then $P \subset G_2$. Thus $\lambda(P) \leq \lambda(G_2)$. Taking sup over all special polygons P gives the desired result.

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(5) Let $P \subset \cup_{k=1}^{\infty} G_k$. Claim: can write $P = \cup_{k=1}^{N} P_{k'}$ with P_k special polygons, $P_k \subset G_{k'}$ and P_k not contained in any other G_k. Then

$$\lambda(P) = \sum_{k=1}^{N} \lambda(P'_k) \leq \sum_{k'=1}^{N} \lambda(G_{k'}) \leq \sum_{k=1}^{\infty} \lambda(G_k).$$

Taking sup over all P, $\lambda\left(\cup_{k=1}^{\infty} G_k\right) \leq \sum_{k=1}^{\infty} \lambda(G_k)$.

(6) Fix N and choose $P_1, \ldots, P_N$ special polygons such that $P_k \subset G_k$. Then P_k's disjoint $\Rightarrow \cup_{k=1}^{N} P_k \subset \cup_{k=1}^{N} G_k \subset \cup_{k=1}^{\infty} G_k$. Thus,

$$\sum_{k=1}^{N} \lambda(P_k) = \lambda\left(\bigcup_{k=1}^{N} P_k\right) \leq \lambda\left(\bigcup_{k=1}^{\infty} G_k\right).$$

Taking sup over all $P_1, \ldots, P_N$, $\sum_{k=1}^{N} \lambda(G_k) \leq \lambda\left(\cup_{k=1}^{\infty} G_k\right)$. Letting $N \to \infty$,

$$\sum_{k=1}^{\infty} \lambda(G_k) \leq \lambda\left(\bigcup_{k=1}^{\infty} G_k\right).$$

The reverse inequality is simply (5), and so equality must hold.

(7) Clearly, for any $\epsilon > 0$ we can find $P' \subset P^\circ$ such that $\lambda(P') > \lambda(P) - \epsilon$. Thus,

$$\lambda(P) - \epsilon < \lambda(P') \leq \lambda(P^\circ),$$

and so $\lambda(P) \leq \lambda(P^\circ)$. Of course, the inequality $\lambda(P^\circ) \leq \lambda(P)$ also holds, because, if $Q \subset P^\circ$ is a special polygon, then $\lambda(Q) \leq \lambda(P)$ and we simply take sup over all such Q.

□

Compact sets. Let $K \subset \mathbb{R}^n$. We will approximate K by open sets. That is, we define

$$\lambda(K) = \inf\{\lambda(G) \mid K \subset G, G \text{ open}\}.$$

Claim: the definition is well-defined. (In particular, a special polygon P is compact.)

Proof. Let $\alpha = $ old $\lambda(P)$ and $\beta = $ new $\lambda(P)$. If $P \subset G$, then $\lambda(P) \leq \lambda(G)$, so by taking inf over all G, $\alpha \leq \beta$. For the other inequality, say $P = \cup_{k=1}^{N} I_k$. Choose I'_k larger than I_k so that $(I'_k)^\circ \supset I_k$ and $\lambda(I'_k) < \lambda(I_k) + \epsilon/N$ for some fixed $\epsilon > 0$. Let $G = \cup_{k=1}^{N} (I'_k)^\circ$. Then

$P \subset G$ and G is open. We have

$$\begin{aligned}\beta \le \lambda(G) &\le \sum_{k=1}^{N} \lambda\left((I_k')^\circ\right)\\ &= \sum_{k=1}^{N} \lambda(I_k')\\ &< \sum_{k=1}^{N} \lambda(I_k) + \epsilon/N\\ &= \alpha + \epsilon.\end{aligned}$$

Since this is true for any $\epsilon > 0$, $\beta \le \alpha$, and consequently $\alpha = \beta$. □

Properties for compact sets.

(1) $0 \le \lambda(K) < \infty$.
(2) $K_1 \subset K_2 \Rightarrow \lambda(K_1) \le \lambda(K_2)$.
(3) $\lambda(K_1 \cup K_2) \le \lambda(K_1) + \lambda(K_2)$.
(4) If K_1 and K_2 are disjoint, then $\lambda(K_1 \cup K_2) = \lambda(K_1) + \lambda(K_2)$.

Proof. (2) If $K_2 \subset G$ (G open) then $K_1 \subset G$.

(3) If $K_1 \subset G_1$ and $K_2 \subset G_2$, then $K_1 \cup K_2 \subset G_1 \cup G_2$. Thus, $\lambda(K_1 \cup K_2) \le \lambda(G_1 \cup G_2) \le \lambda(G_1) + \lambda(G_2)$. Take inf over all $G_1, G_2 \Rightarrow \lambda(K_1 \cup K_2) \le \lambda(K_1) + \lambda(K_2)$.

(4) Since K_1 and K_2 are compact (and disjoint), there exists $\epsilon > 0$ such that an ϵ-neighborhood K_1^ϵ of K_1 does not intersect K_2 and an ϵ-neighborhood K_2^ϵ of K_2 does not intersect K_1. Let G be an open set such that $K_1 \cup K_2 \subset G$. Let $G_1 = G \cap K_1^\epsilon$ and $G_2 = G \cap K_2^\epsilon$. Then G_1 and G_2 are disjoint, $K_i \subset G_i$ for $i = 1, 2$, and

$$\lambda(K_1) + \lambda(K_2) \le \lambda(G_1) + \lambda(G_2) = \lambda(G_1 \cup G_2) \le \lambda(G).$$

Taking inf over all G gives $\lambda(K_1) + \lambda(K_2) \le \lambda(K_1 \cup K_2)$. The reverse inequality is (3) $\Rightarrow \lambda(K_1 \cup K_2) = \lambda(K_1) + \lambda(K_2)$.

□

Inner and outer measure. If $A \subset \mathbb{R}^n$ is arbitrary, then we define both inner and outer measure:

- (Outer measure) $\lambda^*(A) = \inf\{\lambda(G) \mid A \subset G, G \text{ open}\}$.
- (Inner measure) $\lambda_*(A) = \sup\{\lambda(K) \mid K \subset A, K \text{ compact}\}$.

Properties:

(1) $\lambda_*(A) \le \lambda^*(A)$.
(2) $A \subset B \Rightarrow \lambda^*(A) \le \lambda^*(B)$ and $\lambda_*(A) \le \lambda_*(B)$.

(3) $\lambda^*(\cup_{k=1}^\infty A_k \leq \sum_{k=1}^\infty \lambda^*(A_k)$. (Outer measure is countably subadditive.)
(4) If A_k disjoint, then $\lambda_*\left(\cup_{k=1}^\infty A_k\right) \geq \sum_{k=1}^\infty \lambda_*(A_k)$.
(5) If A open or compact, then $\lambda^*(A) = \lambda_*(A) = \lambda(A)$.

Proof. (1) If $K \subset A \subset G$, then $K \subset G$, so $\lambda(K) \leq \lambda(G)$ by definition of λ(compact).

(3) For any $\epsilon > 0$, choose $G_k \supset A_k$ such that $\lambda(G_k) < \lambda^*(A_k) + \epsilon/2^k$. Then

$$\begin{aligned} \lambda^*\left(\bigcup_{k=1}^\infty A_k\right) \leq \lambda\left(\bigcup_{k=1}^\infty G_k\right) &\leq \sum_{k=1}^\infty \lambda(G_k) \\ &< \sum_{k=1}^\infty \left(\lambda^*(A_k) + \epsilon/2^k\right) \\ &= \sum_{k=1}^\infty \lambda^*(A_k) + \epsilon. \end{aligned}$$

(4) Choose $K_k \subset A_k$; then K_k's disjoint. Then

$$\lambda_*\left(\bigcup_{k=1}^\infty A_k\right) \geq \lambda\left(\bigcup_{k=1}^N K_k\right) = \sum_{k=1}^N \lambda(K_k).$$

With N fixed, take sup over all K_k. This gives

$$\lambda_*\left(\bigcup_{k=1}^\infty A_k\right) \geq \sum_{k=1}^\infty \lambda_*(A_k).$$

Letting $N \to \infty$ gives the result.

(5) First let A be open. Then $\lambda^*(A) = \lambda(A)$. If $P \subset A$ with P special polygon, then $\lambda(P) \leq \lambda_*(A)$, which implies that $\lambda(A) \leq \lambda_*(A)$. Thus,

$$\lambda^*(A) = \lambda(A) \leq \lambda_*(A) \leq \lambda^*(A),$$

so all are equal. Now let A be compact. Then $\lambda_*(A) = \lambda(A)$, and $\lambda(A) = \lambda^*(A)$ since the measure of compact sets was defined using open sets.

□