# MEASURE AND INTEGRATION: LECTURE 5

**Definition of $L^1$.** Let $f\colon X \to [-\infty, \infty]$ be measurable. We say that $f$ is in $L^1$ (written $f \in L^1(\mu)$ or simply $f \in L^1$) $\iff \int_X f^+ \, d\mu < \infty$ and $\int_X f^- \, d\mu\infty \iff \int_X |f| \, d\mu < \infty$. Define

$$\int_X f \, d\mu = \int_X f^+ \, d\mu - \int_X f^- \, d\mu$$

when at least one of the terms on the right-hand side is finite.

**Integral of complex functions.** Let $f\colon X \to \mathbb{C}$ be measurable. That is, $f = u + iv$ where $u, v\colon X \to \mathbb{R}$ are measurable. Then

$$\begin{aligned} f \in L^1(\mu) &\iff \int_X |F| \, d\mu < \infty \\ &\iff \int_X |u| \, d\mu < \infty \text{ and } \int_X |v| \, d\mu < \infty. \end{aligned}$$

Define

$$\begin{aligned} \int_X f \, d\mu &= \int_X u \, d\mu + i \int_X v \, d\mu \\ &= \int_X u^+ \, d\mu - \int_X u^- \, d\mu + i \int_X v^+ \, d\mu - i \int_X v^- \, d\mu. \end{aligned}$$

**Theorem 0.1.** *Let $f, g \in L^1(\mu)$. If $\alpha, \beta \in \mathbb{C}$, then $\alpha f + \beta g \in L^1(\mu)$ and*

$$\int_X (\alpha f + \beta g) d\mu = \alpha \int_X f \, d\mu + \beta \int_X g \, d\mu.$$

*Proof.* First, $\alpha f + \beta g$ is measurable, and by the triangle inequality, $|\alpha f + \beta g| \le |\alpha| \, |f| + |\beta| \, |g|$, so

$$\int_X |\alpha f + \beta g| \le |\alpha| \int_X |f| + |\beta| \int_X |g| < \infty.$$

Just need to show that

(1) $\int_X (f + g) d\mu = \int_X f \, d\mu + \int_X g \, d\mu$, and
(2) $\int_X (\alpha f) d\mu = \alpha \int_X f \, d\mu$.

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For (1), assume $f, g$ real; the complex case follows from the real case. Let $h = f + g$. Then $h^+ - h^- = f^+ - f^- + g^+ - g^-$, so $h^+ + f^- + g^- = f^+ + g^+ + h^-$. Since the integral is linear for non-negative functions,

$$\int h^+ + \int f^- + \int g^- = \int f^+ + \int g^+ + \int h^- \Rightarrow$$

$$\int h^+ - \int h^- = \int f^+ - \int f^- + \int g^+ - \int g^-.$$

Thus, $\int f + g = \int f + \int g$.

For (2), let $\alpha = a + bi$ for $a, b \in \mathbb{R}$. Then

$$\begin{aligned}\int \alpha f = \int (a+bi)(u+iv) &= \int au + aiv + biu - bv \\ &= \int (au - bv + i(av + bu) \\ &= \int (au - bv) + i \int (av + bu).\end{aligned}$$

Also,

$$\begin{aligned}(a+bi)\int (u+iv) &= (a+bi)\left(\int u + i\int v\right) \\ &= a\int u + bi\int u + ai\int v - b\int v.\end{aligned}$$

So, just need to show that $\int au = a\int u$. If $a = 0$, then both sides vanish. If $a > 0$, then

$$\begin{aligned}\int (au) &= \int (au)^+ - \int (au)^- \\ &= \int a \cdot u^+ - \int a \cdot u^- \\ &= a\int u^+ - a\int u^- = a\int u.\end{aligned}$$

If $a < 0$, then

$$\begin{aligned}\int au &= \int (au)^+ - \int (au)^- \\ &= \int (-a) \cdot u^- - \int (-a) \cdot u^+ \\ &= -a\int u^- - (-a)\int u^+ \\ &= a\left(\int u^+ - \int u^-\right) = a\int u.\end{aligned}$$

□

**Theorem 0.2.** *If $f \in L^1(\mu)$, then*

$$\left| \int_X f \; d\mu \right| \leq \int_X |f| \, d\mu.$$

*Proof.* For some $\theta \in [0, 2\pi)$,

$$\int_X f \; d\mu = \left| \int_X f \; d\mu \right| e^{i\theta}.$$

Hence,

$$\begin{aligned} \left| \int_X f \; d\mu \right| = e^{-i\theta} \int_X f \; d\mu &= \int_X (e^{-i\theta} f) d\mu \\ &= \mathrm{Re} \left( \int_X e^{-i\theta} f \right) d\mu \\ &= \int_X \mathrm{Re}(e^{-i\theta} f) d\mu \\ &\leq \int_X \left| e^{-i\theta} f \right| d\mu = \int_X |f| \, d\mu. \end{aligned}$$

□

**Dominated convergence.**

**Theorem 0.3.** *Let $f_n \colon X \to \mathbb{C}$ be a sequence of measurable functions, and assume that $f(x) = \lim_{n\to\infty} f_n(x)$ (that is, the sequence $f_n$ converges pointwise). If there exists $g \in L^1(\mu)$ such that $|f_n(x)| \leq g(x)$ for all $n$ and for all $x \in X$, then $f \in L^1(\mu)$ and*

$$\lim_{n\to\infty} \int_X |f_n - f| \, d\mu = 0, \;\; \textit{so} \;\; \lim_{n\to\infty} \int_X f_n \; d\mu = \int_X f \; d\mu.$$

*Proof.* Since $|f_n(x)| \leq g(x)$ for all $n$, the limit $f$ has the property that $|f| \leq |g(x)|$. This means that $\int |f| \leq \int |g| < \infty$, so $f \in L^1(\mu)$. Next, $|f_n - f| \leq |f_n| + |f| \leq 2g$, which means that $2g - |f_n - f| \geq 0$. Applying Fatou's lemma,

$$\begin{aligned} \int_X 2g \; d\mu &\leq \liminf \int_X (2g - |f_n - f|) d\mu \\ &= \int_X 2g \; d\mu + \liminf \int_X - |f_n - f| \, d\mu \\ &= \int_X 2g \; d\mu - \limsup \int_X |f_n - f| \, d\mu. \end{aligned}$$

Since $\int_X 2g < \infty$, it can be cancelled from both sides. Thus,

$$\limsup \int_X |f_n - f|\, d\mu \leq 0,$$

and so

$$\lim \int_X |f_n - f|\, d\mu = 0.$$

From the previous theorem,

$$\begin{aligned} &\left|\int_X (f_n - f) d\mu\right| \leq \int_X |f_n - f|\, d\mu \\ \Rightarrow &\left|\int_X f_n\ d\mu - \int_X f\ d\mu\right| \leq \int_X |f_n - f|\, d\mu \to 0 \\ \Rightarrow &\int_X f_n\ d\mu \to \int_X f\ d\mu. \end{aligned}$$

□

**Sets of measure zero.** Let $(X, \mathcal{M}, \mu)$ be a measure space and $E \in \mathcal{M}$. A set $E$ has measure zero if and only if $\mu(E) = 0$. If $f, g\colon X \to \mathbb{C}$, then $f = g$ almost everywhere (a.e.) if $N = \{x \mid f(x) = g(x)\}$ has measure zero. Define an equivalence relation $f \sim g$ if $f = g$ a.e.

**Proposition 0.4.** *If $f \sim g$, then, for all $E \in \mathcal{M}$, $\int_E f\ d\mu = \int_E g\ d\mu$.*

*Proof.* Write $E$ as disjoint union $E = (E \setminus N) \cup (E \cap N)$. Then, since $f = g$ away from $N$, and since $N$ has measure zero,

$$\begin{aligned} \int_E f\ d\mu &= \int_{E\setminus N} f\ d\mu + \int_{E\cap N} f\ d\mu \\ &= \int_{E\setminus N} g\ d\mu + 0 = \int_E g\ d\mu. \end{aligned}$$

□

**Completion of a $\sigma$-algebra.**

**Theorem 0.5.** *Let $(X, \mathcal{M}, \mu)$ be a measure space. Let*

$$\mathcal{M}^* = \{E \subset X \mid \exists A, B \in \mathcal{M}\colon A \subset E \subset B\ \&\ \mu(B \setminus A) = 0\}.$$

*Now define $\mu(E) = \mu(A)$ for all $E \in \mathcal{M}^*$. Then $\mathcal{M}^*$ is a $\sigma$-algebra and this definition of $\mu$ is a measure.*

The measure space $(X, \mathcal{M}^*, \mu)$ is a called the *completion* of the measure space $(X, \mathcal{M}, \mu)$. A measure space is *complete* if it is equal to its completion.

**Note.** If $f$ is only defined a.e. (say, except for a set $N$ of measure zero), then we can define $f(x) = 0$ for all $x \in N$. $\Rightarrow \int f$ is well defined.

**Theorem 0.6.** (a) *Let* $f : X \to [0, \infty]$ *be measurable,* $E \in \mathcal{M}$*, and* $\int_E f\ d\mu = 0$*. Then* $f = 0$ *a.e. on* $E$*.*

(b) *Let* $f \in L^1(\mu)$ *and* $\int_E f\ d\mu = 0$ *for every* $E \in \mathcal{M}$*. Then* $f = 0$ *a.e. on* $X$*.*

*Proof.* (a) Let $A_n = \{x \in E \mid f(x) > 1/n\}$. Then

$$\int_E f\ d\mu \geq \int_{A_n} f\ d\mu \geq \int_{A_n} 1/n\ d\mu = \frac{1}{n}\mu(A_n),$$

which implies that $\mu(A_n) = 0$. But $\{x \mid f(x) > 0\} = \cup_{i=1}^{\infty} A_n$ and $\mu(\{x \mid f(x) > 0\}) \leq \sum_{i=1}^{\infty} \mu(A_n) = 0$.

(b) Let $f = u + iv$. Choose $E = \{x \mid u(x) \geq 0\}$. Then $\int_E f = \int_E u^+ + i \int_E v \Rightarrow \int_E u^+ = 0$ and by (a), $u^+ = 0$ a.e.

□

**Theorem 0.7.** *Let* $E_k \in \mathcal{M}$ *such that* $\sum_{k=1}^{\infty} \mu(E_k) < \infty$*. Then almost every* $x \in X$ *lie in at most finitely many* $E_k$*.*

*Proof.* Let $A = \{x \in X \mid x \in E_k$ for infinitely many $k\}$. NTS $\mu(A) = 0$. Let $g = \sum_{i=1}^{\infty} \chi_{E_k}$. Then $x \in A \iff g(x) = \infty$. We have

$$\int_X g\ d\mu = \sum_{i=1}^{\infty} \int_X \chi_{E_k}\ d\mu = \sum_{i=1}^{\infty} \mu(E_k) < \infty.$$

In other words, $g \in L^1(\mu)$ and thus $g(x) < \infty$ a.e. □