MEASURE AND INTEGRATION: LECTURE 4

Integral is additive for simple functions.

Proposition 0.1. *Let s and t be non-negative measurable simple functions. Then $\int_X (s+t)d\mu = \int_X s\ d\mu + \int_X t\ d\mu$.*

Proof. Let $E \in \mathcal{M}$ and define $\varphi(E) = \int_E s\ d\mu$. First we show that φ is measurable. To this end, let $E_i \in \mathcal{M}$ with E_i disjoint and $E = \cup_{i=1}^{\infty} E_i$. Then

$$\begin{aligned}\varphi(E) &= \sum_{i=1}^{N} \alpha_i \mu(A_i \cap E) = \sum_{i=1}^{N} \alpha_i \mu\left(A_i \cap (\cup_{j=1}^{\infty} E_j)\right) \\ &= \sum_{i=1}^{N} \alpha_i \mu\left(\cup_{i=1}^{\infty} (A_i \cap E_j)\right) = \sum_{i=1}^{N} \alpha_i \sum_{j=1}^{\infty} \mu\left(A_i \cap E_j\right) \\ &= \sum_{j=1}^{\infty} \sum_{i=1}^{N} \alpha_i \mu(A_i \cap E_j) = \sum_{j=1}^{\infty} \int_{E_j} s\ d\mu = \sum_{j=1}^{\infty} \varphi(E_j)\end{aligned}$$

Let $s = \sum_{i=1}^{N} \alpha_i \chi_{A_i}$ and $t = \sum_{j=1}^{M} \beta_j \chi_{B_j}$. Let $E_{ij} = A_i \cap B_j$. Then

$$\int_{E_{ij}} (s+t)d\mu = (\alpha_i + \beta_j)\mu(E_{ij})$$

and

$$\int_{E_{ij}} s\ d\mu + \int_{E_{ij}} t\ d\mu = \alpha_i \mu(E_{ij}) + \beta_j \mu(E_{ij})$$

so that

$$\int_{E_{ij}} s\ d\mu + \int_{E_{ij}} t\ d\mu = \int_{E_{ij}} (s+t)d\mu.$$

Write $X = \cup_{i,j} E_{ij}$ as a disjoint union. Then

$$\int_X s\ d\mu + \int_X t\ d\mu = \int_X (s+t)d\mu.$$

□

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Next, we want to prove

$$\int_X (f+g)d\mu = \int_X f\ d\mu + \int_X g\ d\mu.$$

If $s \leq f$ simple and $t \leq g$ simple, then $s+t \leq f+g$ simple. The only thing we know so far is that

$$\int_X (f+g)d\mu \geq \int_X f\ d\mu + \int_X g\ d\mu.$$

One way to obtain equality is to define an upper integral and a lower integral, and say that a function is integrable $\iff$ its upper and lower integral are equal and finite. Then we should prove that f integrable $\iff$ f measurable. But this is not necessary, and we will use the definition we have.

Monotone convergence.

Theorem 0.2. *Let $f_n : X \to [0,\infty]$ be a sequence of measurable functions such that*

(a) $0 \leq f_1 \leq f_2 \leq \cdots \leq \infty$, *and*
(b) $f_n(x) \to f(x)$ *as* $n \to \infty$ *for all* $x \in X$.

Then $\int_X f_n d\mu \to \int_X f d\mu$ *as* $n \to \infty$.

Proof. Since $f_i \leq f_{i+1}$ for all i, we have $\int f_i \leq \int f_{i+1}$. Thus $\int f_i \to \alpha$ for some $\alpha \in [0,\infty]$. Also $f_n \leq f \Rightarrow \int f_i \leq \int f$, so $\alpha \leq \int f$. Next, let s be simple and measurable with $0 \leq s \leq f$ and let c be a constant such that $0 \leq c \leq 1$. Define $E_n = \{x \mid f_n(x) \geq cs(x)\}$ for $n = 1, 2, \ldots$. Then E_i is measurable and $E_1 \subset E_2 \subset \cdots$, and $X = \cup_{i=1}^{\infty} E_i$. Indeed, if $f(x) = 0$ for any $x \in X$, then $x \in E_1$, and if $f(x) > 0$, then $cs(x) < f(x)$. Since $f_n \to f$, $f_n > cs(x)$ for n large; thus $x \in E_n$ for n large.

Lastly,

$$\int_X f_n\ d\mu \geq \int_{E_n} f_n\ d\mu \geq c \int_{E_n} s\ d\mu.$$

Letting $n \to \infty$,

$$\alpha \geq c \lim_{n\to\infty} \int_{E_n} s\ d\mu.$$

Thus $\alpha \geq c \int_X s d\mu$ for any $c < 1$. Let $c \to 1$. Then $\alpha \geq \int_X s\ d\mu$ for any simple, measurable $0 \leq s \leq f$. We conclude that

$$\alpha = \int_X f\ d\mu = \lim_{n\to\infty} f_n\ d\mu.$$

□

Integral is additive for non-negative measurable functions.

Theorem 0.3. *Let* $f_n\colon X \to [0,\infty]$ *be a sequence of measurable functions and* $f(x) = \sum_{n=1}^{\infty} f_n(x)$*. Then* $\int_X f\ d\mu = \sum_{n=1}^{\infty} \int_X f_n\ d\mu$*.*

Proof. First, claim if f, g measurable, then $\int_X f+g = \int_X f + \int_X g$. Let $0 \le s_1 \le s_2 \le \cdots$ be simple, measurable, and $s_i \to f$. Similarly, let $0 \le t_1 \le t_2 \le \cdots$ be simple, measurable, and $t_i \to g$. Then $s_i + t_i$ are simple and $s_i + t_i \to f + g$, which implies that $\int_X (s_i + t_i) d\mu = \int_X s_i\ d\mu + \int_X t_i\ d\mu$. By the monotone convergence theorem, the claim is proved. Using induction, $\int_X \left(\sum_{i=1}^N f_i\right) d\mu = \sum_{i=1}^N \int_X f_i\ d\mu$.

Let $g_N = \sum_{i=1}^N f_i$. Then $g_N \to \sum_{n=1}^{\infty} f_n = f(x)$ as $N \to \infty$ monotonically. Thus

$$\begin{aligned}
\int_X \sum_{n=1}^{\infty} f_n\ d\mu &= \lim_{N\to\infty} \int_X g_N\ d\mu \\
&= \lim_{N\to\infty} \int_X \sum_{i=1}^{N} f_i\ d\mu \\
&= \lim_{N\to\infty} \sum_{i=1}^{N} \int_X f_i\ d\mu \\
&= \sum_{i=1}^{\infty} \int_X f_i\ d\mu
\end{aligned}$$

□

Interchanging summation and integration.

Corollary 0.4. *Let* $X = \mathbb{Z}^+ \equiv \{1, 2, 3, \ldots\}$ *and* μ *be the counting measure. Let* $a_{ij} \ge 0$ *and* $f_j = a_{ij}\colon \mathbb{Z}^+ \to [0,\infty]$*. Then*

$$\int \sum_{j=1}^{\infty} f_j = \sum_{j=1}^{\infty} \int f_j,$$

so that

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}.$$

Fatou's Lemma.

Lemma 0.5. *Let* $f_n\colon X \to [0,\infty]$ *be a sequence of measurable functions. Then*

$$\int_X \left(\liminf_{n\to\infty} f_n\right) d\mu \le \liminf_{n\to\infty} \int_X f_n\ d\mu.$$

Proof. Let $g_k(x) = \inf_{i \geq k} f_i(x)$. Then $g_1 \leq g_2 \leq \cdots$ and

$$\lim_{k\to\infty} g_k = \liminf_{k\to\infty} f_k.$$

Also, $g_k \leq f_k$, so monotone convergence implies that

$$\begin{aligned}
\int_X \liminf f_k \; d\mu &= \int_X \lim g_k \; d\mu \\
&= \lim \int_X g_k \; d\mu \\
&= \liminf \int_X g_k \; d\mu \\
&\leq \liminf \int_X f_k \; d\mu.
\end{aligned}$$

□