MEASURE AND INTEGRATION: LECTURE 3

Riemann integral. If s is simple and measurable then

$$\int_X s d\mu = \sum_{i=1}^N \alpha_i \mu(E_i),$$

where $s = \sum_{i=1}^N \alpha_i \chi_{E_i}$. If $f \geq 0$, then

$$\int_X f d\mu = \sup \left\{ \int_X s d\mu \mid 0 \leq s \leq f, s \text{ simple \& measurable} \right\}.$$

Recall the Riemann integral of function f on interval $[a, b]$. Define lower and upper integrals $L(f, P)$ and $U(f, P)$, where P is a partition of $[a, b]$. Set

$$\underline{\int} f = \sup_P L(f, P) \quad \text{and} \quad \overline{\int} f = \inf_P U(f, P).$$

A function f is Riemann integrable $\iff$

$$\underline{\int} f = \overline{\int} f,$$

in which case this common value is $\int f$.

A set $B \subset \mathbb{R}$ has *measure zero* if, for any $\epsilon > 0$, there exists a countable collection of intervals $\{I_i\}_{i=1}^\infty$ such that $B \subset \cup_{i=1}^\infty I_i$ and $\sum_{i=1}^\infty \lambda(I_i) < \epsilon$. Examples: finite sets, countable sets. There are also uncountable sets with measure zero. However, any interval does not have measure zero.

Theorem 0.1. *A function f is Riemann integrable if and only if f is discontinuous on a set of measure zero.*

A function is said to have a property (e.g., continuous) *almost everywhere* (abbreviated a.e.) if the set on which the property does not hold has measure zero. Thus, the statement of the theorem is that f is Riemann integrable if and only if it is continuous almost everywhere.

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Recall positive measure: a measure function $\mu\colon \mathcal{M} \to [0,\infty]$ such that $\mu\left(\cup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$ for $E_i \in \mathcal{M}$ disjoint.

Examples.

(1) "Counting measure." Let X be any set and $\mathcal{M} = \mathcal{P}(X)$ the set of all subsets. If $E \subset X$ is finite, then $\mu(E) = \#E$ (the number of elements in E). If $E \subset X$ is infinite, then $\mu(E) = \infty$.

(2) "Unit mass at x_0 – Dirac delta function." Again let X be any set and $\mathcal{M} = \mathcal{P}(X)$. Choose $x_0 \in X$. Set

$$\mu(E) = \begin{cases} 1 & \text{if } x_0 \in E; \\ 0 & \text{if } x_0 \notin E. \end{cases}$$

Theorem 0.2. (1) *If $E \subset \mathbb{R}$ and $\mu(E) < \infty$, then $\mu(\emptyset) = 0$.*

(2) *(Monotonicity) $A \subset B \Rightarrow \mu(A) \leq \mu(B)$.*

(3) *If $A_i \in \mathcal{M}$ for $i = 1, 2, \ldots$, $A_1 \subset A_2 \subset \cdots$, and $A = \cup_{i=1}^{\infty} A_i$, then $\mu(A_i) \to \mu(A)$ as $i \to \infty$.*

(4) *If $A_i \in \mathcal{M}$ for $i = 1, 2, \ldots$, $A_1 \supset A_2 \supset \cdots$, $\mu(A_1) < \infty$, and $A = \cap_{i=1}^{\infty} A_i$, then $\mu(A_i) \to \mu(A)$ as $i \to \infty$.*

Proof. (1) $E = E \cup \emptyset \Rightarrow \mu(E) = \mu(E) + \mu(\emptyset)$.

(2) $B = A \cup (B \setminus A) \Rightarrow \mu(B) = \mu(A) + \mu(B \setminus A) \geq \mu(A)$.

(3) Let $B_1 = A_1$, $B_2 = A_2 \setminus A_1$, $B_3 = A_3 \setminus A_2$, ... Then the B_i are disjoint, $A_n = B_1 \cup \cdots \cup B_n$, and $A = \cup_{i=1}^{\infty} B_i$. Thus, $\mu(A_n) = \mu(B_1) + \cdots + \mu(B_n) = \sum_{i=1}^{n} \mu(B_i)$, and (3) follows.

(4) Let $C_n = A_1 \setminus A_n$. Then $C_1 \subset C_2 \subset \cdots$. We have $\lambda(C_n) = \lambda(A_1) - \lambda(A_n)$. Also, $A_1 \setminus A = \cup_n C_n$. Thus, $A_1 \cap (\cap A_i)^c = \cup (A_1 \setminus A_n)$, and so

$$\mu(A_1 \setminus A) = \lim_{i \to \infty} \mu(C_i) = \mu(A_1) - \lim_{i \to \infty} \mu(A_n).$$

Hence, $\mu(A_1) \to \mu(A)$.

□

Properties of the Integral.

(a) If $0 \leq f \leq g$ on E, then $\int_E f d\mu \leq \int_E g d\mu$.

(b) If $A \subset B$, $A, B \in \mathcal{M}$, and $f \geq 0$, then $\int_A f d\mu \leq \int_B f d\mu$.

(c) If $f \geq 0$ and $c \in [0, \infty)$ is a non-negative constant, then $\int_E c f d\mu = c \int_E f d\mu$.

(d) If $f(x) = 0$ for all $x \in E$, then $\int_E f d\mu = 0$.

(e) If $\mu(E) = 0$, then $\int_E f d\mu = 0$.

(f) If $f \geq 0$, then $\int_E f d\mu = \int_E \chi_E f d\mu$.

Proof. (a) If $s \leq f$ is simple, then $s \leq g$ so the sup on g is over a larger class of simple functions than the sup on f.

(b) We have $E_i \cap A \subset E_i \cap B$ for all E_i. If s is simple,

$$\int_A s d\mu = \sum_{i=1}^{N} \alpha_i \mu(E_i \cap A) \leq \sum_{i=1}^{N} \alpha_i \mu(E_i \cap B) = \int_B s d\mu.$$

(c) For any simple s, $\int_E c s d\mu = c \int_E s d\mu$ since

$$\sum_i (c\alpha_i)\chi_{E_i} = c \sum_i \alpha_i \chi_{E_i}.$$

For any constant c, $s \leq f \iff cs \leq cf$. Thus,

$$\int cf = \sup_{s \leq cf} \int s = \sup_{s/c \leq f} \int s = \sup_{s' f} \int cs' = c \int f.$$

(d) Let $s \leq f$ be simple and $s = \sum_{i=1}^{N} \alpha_i \chi_{E_i}$. Without loss of generality, $\alpha_1 = 0$ and $E_1 \supset E$. Thus,

$$\int_E s d\mu = \sum_{i=1}^{N} \alpha_i \mu(E_i \cap E) = \alpha_1 \mu(E) = 0.$$

(The convention here and throughout is that $0 \cdot \infty = 0$.)

(e) If $s \leq f$ and $s = \sum_i \alpha_i \chi_{E_i}$, then $\int_E s = \sum_i \alpha_i \mu(E \cap E_i) = 0$.

(f) This could have been the definition of the integral.

$\square$