MEASURE AND INTEGRATION: LECTURE 24

Inequalities

Generalized Minkowski inequality. Let $\mathbb{R}^n = \mathbb{R}^\ell \times \mathbb{R}^m$ and $z = (x, y) \in \mathbb{R}^n$. If $\mathbb{R}^n \to \mathbb{C}$ is measurable, then

$$\int_{\mathbb{R}^\ell} |f(x,y)|^p \ dx \colon \mathbb{R}^m \to \mathbb{R} = \|f_y\|_{L^p(\mathbb{R}^\ell)} : \mathbb{R}^m \to \mathbb{R}$$

is $\mathbb{R}^n$-measurable for $1 \le p < \infty$.

Assume that

$$\int_{\mathbb{R}^m} \|f_y\|_{L^p(\mathbb{R}^\ell)} \ dy < \infty.$$

Then for a.e. $x \in \mathbb{R}^\ell$, $f_x(y) \colon \mathbb{R}^m \to \mathbb{C}$ is in $L^1(\mathbb{R}^m)$. Let

$$F(x) = \int_{\mathbb{R}^m} f_x(y) \ dy.$$

Then $F(x) \colon \mathbb{R}^\ell \to \mathbb{C}$ is $\mathbb{R}^\ell$-measurable and we have

$$\|F\|_{L^p(\mathbb{R}^\ell)} \le \int_{\mathbb{R}^m} \|f_y\|_{L^p(\mathbb{R}^\ell)} \ dy.$$

Note this is

$$\left(\int_{\mathbb{R}^\ell} \left| \int_{\mathbb{R}^m} f(x,y) \ dy \right|^p \ dx \right)^{1/p} \le \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^\ell} |f(x,y)|^p \ dx \right)^{1/p} \ dy.$$

We could replace by X, Y σ-finite measure space and $Y = \{p_1, \dots, p_n\}$, dy the counting measure and get old Minkowski's.

Proof. We have

$$|F(x)| \le \int_{\mathbb{R}^m} |f_x(y)| \ dy,$$

so without loss of generality, assume $f \ge 0$. If $p = 1$, then Fubini's theorem applies; so now let $p > 1$.

Define $g \colon \mathbb{R}^\ell \times \mathbb{R}^m \to \mathbb{R}^{\ge 0}$ by

$$g(x,y) = \begin{cases} f(x,y) \, \|f_y\|_{L^p(\mathbb{R}^\ell)}^{-1/p'} & \text{if } 0 < \|f_y\|_p < \infty; \\ 0 & \text{if } \|f_y\|_p = 0; \\ \infty & \text{if } \|f_y\|_p = \infty. \end{cases}$$

Then, for each y,

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(1) $f(x,y) \le g(x,y) \|f_y\|_p^{1/p'}$ for a.e. x, and
(2)

$$\|g_y\|_{L^p(\mathbb{R}^\ell)} = \|f_y\|_p^{1/p}.$$

Then

$$\begin{aligned} F(x) &= \int_{\mathbb{R}^m} f(x,y)\ dy \\ &\le \int_{\mathbb{R}^m} g(x,y) \|f_y\|_p^{1/p'}\ dy \\ &\le \|g_x\|_{L^p(\mathbb{R}^m)} \left(\int \|f_y\|_p\ dy \right)^{1/p'} \\ &= \|g_x\|_{L^p(\mathbb{R}^m)} \cdot C^{1/p'}, \end{aligned}$$

where $C = \int_{\mathbb{R}^m} \|f_y\|_p\ dy$.

We now use Fubini's theorem:

$$\begin{aligned} \|F(x)\|_{L^p(\mathbb{R}^\ell)}^p &\le C^{p/p'} \int_{\mathbb{R}^\ell} \left(\|g_x\|_{L^p(\mathbb{R}^m)}^p \right)\ dx \\ &= c^{p-1} \int_{\mathbb{R}^\ell} \left(\int_{\mathbb{R}^m} g(x,y)^p\ dy \right)\ dx \\ &= c^{p-1} \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^\ell} g(x,y)^p\ dx \right)\ dy \\ &= c^{p-1} \int_{\mathbb{R}^m} \|g_y\|_{L^p(\mathbb{R}^\ell)}^p\ dy \\ &= c^{p-1} \int_{\mathbb{R}^m} \|f_y\|_{L^p(\mathbb{R}^\ell)}\ dy \\ &= c^{p-1} c = c^p = \int_{\mathbb{R}^m} \|f_y\|_{L^p(\mathbb{R}^\ell)}\ dy. \end{aligned}$$

Thus,

$$\|F(x)\|_{L^p(\mathbb{R}^\ell)} \le \int_{\mathbb{R}^m} \|f_y\|_{L^p(\mathbb{R}^\ell)}\ dy$$

and so

$$\left\| \int_{\mathbb{R}^m} f(x,y)\ dy \right\|_{L^p(\mathbb{R}^\ell)} \le \int_{\mathbb{R}^m} \|f_y\|_{L^p(\mathbb{R}^\ell)}\ dy.$$

□

Application. Let $f \in L^1(\mathbb{R}^n$ and $g \in L^p(\mathbb{R}^n)$. Then $f * g \in L^p(\mathbb{R}^n)$ since

$$\begin{aligned}
&\left(\int_{\mathbb{R}^n} \left|\int_{\mathbb{R}^n} f(y)g(x-y)\ dy\right|^p\ dx\right)^{1/p} \\
&\leq \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(y)g(x-y)|^p\ dx\right)^{1/p}\ dy \\
&= \int_{\mathbb{R}^n} |f(y)| \left(\int_{\mathbb{R}^n} |g(x-y)|^p\ dx\right)^{1/p}\ dy \\
&= \int_{\mathbb{R}^n} |f(y)|\, \|g\|_p\ dy \\
&= \|f\|_1 \|g\|_p .
\end{aligned}$$

Distribution functions. Suppose $f\colon X \to [0,\infty]$ and let $\mu\{f > t\} = \mu(x \mid f(x) > t\}$.

Theorem 0.1.

$$\int_X f\ d\mu = \int_0^\infty \mu\{f > t\}\ dt$$

and

$$\int_X f^p\ d\mu = p\int_0^\infty \mu\{f > t\}t^{p-1}\ dt$$

More generally, if φ *is differentiable, then*

$$\int_X \varphi \circ f\ d\mu = \int_0^\infty \mu\{f > t\}\varphi'(t)\ dt.$$

Proof. We have

$$\begin{aligned}
\int_{\mathbb{R}^n} |f|\ dx &= \int_{\mathbb{R}^n} \left(\int_0^{|f(x)|}\ dt\right)\ dx \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}} \chi_{[0,f(x)]}(t)\ dt\ dx \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}^n} \chi_{[0,f(x)]}(t)\ dx\ dt
\end{aligned}$$

Then

$$\begin{aligned}
\int_{\mathbb{R}^n} |f|^p\ dx &= \int_0^\infty \mu\{|f|^p > t\}\ dt \\
&= \int_0^\infty \mu\{|t| > t^{1/p}\}\ dt \\
&= p\int_0^\infty \mu\{|f| > s\}s^{p-1}\ ds,
\end{aligned}$$

letting $s = t^{1/p}$, so $s^p = t$ and $dt = ps^{p-1}ds$. □

Marcinkiewicz interpolation. Recall the maximal function

$$Mf(x) = \sup_{0<r<\infty} \frac{1}{\lambda(B(x,r))} \int_{B(x,r)} |f(y)| \ dy.$$

Note if $f \in L^\infty$, then $\|Mf\|_\infty \leq \|f\|_\infty$. Thus, M maps L^∞ into itself: $M\colon L^\infty \to L^\infty$.

On the other hand, by Hardy-Littlewood, if $f \in L^1$, then

$$\mu\{Mf > t\} \leq \frac{3^n}{t} \|f\|_1 , \tag{0.1}$$

and M maps L^1 to weak L^1.

Using a method called Marcinkiewicz interpolation, we prove the following.

Theorem 0.2. *Let $1 < p < \infty$ and $f \in L^p$. Then $Mf \in L^p$, and*

$$\|Mf\|_p \leq C(n,p) \|f\|_p , \tag{0.2}$$

where $C(n,p)$ is bounded as $p \to \infty$ and $C(n,p) \to \infty$ as $p \to 1$.

Proof. Observe that $Mf = M|f|$, so assume $f \geq 0$. Choose a constant $0 < c < 1$ (we will choose the best c later). For $t \in (0,\infty)$, write $f = g_t + h_t$, where

$$g_t(x) = \begin{cases} f(x) & f(x) > ct; \\ 0 & f(x) \leq ct. \end{cases}$$

So, $0 \leq h_t(x) \leq ct$ for every x, and thus $h_t \in L^\infty$. We have

$$Mf \leq Mg_t + Mh_t \leq Mg_t + ct$$

from (0.2). Thus, $Mf - ct \leq Mg_t$, o if $Mf(x) > t$, then $(1-c)t \leq Mg_t(x)$.

Let $E_t = \{f > ct\}$. Then

$$\begin{aligned} \lambda\{Mf > t\} &\leq \lambda\{Mg_t > (1-c)t\} \\ &\leq \frac{3^n}{(1-c)t} \|g_t\|_1 \qquad \text{from (0.1)} \\ &= \frac{3^n}{(1-c)t} \int_{E_t} f \ dx. \end{aligned}$$

Thus,

$$\begin{aligned}\int_{\mathbb{R}^n} (Mf)^p \; dx &= p \int_0^\infty \lambda\{Mf > t\} t^{p-1} \; dt \\ &\leq \frac{3^n p}{1-c} \int_0^\infty t^{p-2} \left(\int_{E_t} f \; dx \right) \; dt \\ &= \frac{3^n p}{1-c} \int_{\mathbb{R}^n} \left(f(x) \int_0^{f(x)/c} t^{p-2} \; dt \right) \; dx \\ &= \frac{3^n p}{1-c} \int_{\mathbb{R}^n} f(x) \frac{1}{p-1} \left(\frac{f(x)}{c} \right)^{p-1} \; dx \\ &= \frac{3^n p}{1-c} \cdot \frac{c^{1-p}}{p-1} \int_{\mathbb{R}^n} f(x)^p \; dx \\ &= C(n,p) \, \|f\|_p^p .\end{aligned}$$

Thus,

$$\|Mf\|_p \leq \underbrace{\frac{3^n p c^{1-p}}{(1-c)(p-1)}^{1/p}}_{\to 1 \text{ as } p \to \infty} \|f\|_p .$$

Choose $c = 1/p' = (p-1)/p$; this gives the best constant. □