MEASURE AND INTEGRATION: LECTURE 21

Approximations. Let

$$h(t) = \begin{cases} 0 & t \le 0; \\ \exp(-1/t) & t > 0. \end{cases}$$

Then $h \in C^\infty$ (infinitely differentiable with continuous derivatives). Define $\phi\colon \mathbb{R}^n \to \mathbb{R}$ by $\phi(x_1, \ldots, x_n) = h(1 - |x|^2)$. If $|x|^2 > 1$, then $1 - |x|^2 < 0 \Rightarrow \phi = 0$ on $B(0,1)^c$. Thus, $\phi \in C_c^\infty(\mathbb{R}^n)$. Redefine ϕ so that $\int_{\mathbb{R}^n} \phi \; dz = 1$.

Now define $\phi_a(x) = a^{-n}\phi(x/a)$. Then ϕ_a supported on a ball of radius a and

$$\int_{\mathbb{R}^n} \phi_a(x) dx = 1$$

by a linear change of variables.

Given f, define $f_a(x) = f * \phi_a = \int_{\mathbb{R}^n} f(y)\phi_a(x-y) \; dy$. Then $f_a(x) \in C_0^\infty$ since

$$\frac{\partial^{(k)}}{\partial x^{(k)}} f_a(x) = \int_{\mathbb{R}^n} f(y) \frac{\partial^{(k)}}{\partial x^{(k)}} \phi_a(x-y) \; dy,$$

and if f has compact support, then so does f_a.

Suppose $f \in L^1(\mathbb{R}^n)$ and define

$$g(x) = \int_{R^n} f(y)\phi_a(x-y) \; dy = f * \phi_a.$$

Note that $\phi_a(x-y)$ is bounded and the integrand is integrable.

Lemma 0.1. *The function $g(x)$ is continuous.*

Proof. Fix x_0. Then

$$\lim_{x \to x_0} g(x) = \lim_{x \to x_0} \int_{\mathbb{R}^n} f(y)\phi_a(x-y) \; dy,$$

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and since $f(y)\phi_a(x-y) \le C\,|f(y)| \in L^1$, we may apply LDCT so that the RHS above equals

$$\begin{aligned} &= \int_{\mathbb{R}^n} \lim_{x\to x_0} f(y)\phi_a(x-y)\ dy \\ &= \int_{\mathbb{R}^n} f(y)\phi_a(x-y)\ dy \quad \text{since } \phi_a \in C_c \\ &= g(x_0). \end{aligned}$$

□

Lemma 0.2. *The kth partial derivatives of g exist and are continuous for $k = 1, 2, \ldots$. In other words, $g \in C^\infty$.*

Proof. Let $e_k = (0, \ldots, 0, 1, 0, \ldots, 0)$, the vector whose kth coordinate is equal to 1 and all other coordinates are zero. We have

$$\begin{aligned} \frac{g(x+te_k)-g(x)}{t} &= \frac{\int_{\mathbb{R}^n} f(y)\phi_a(x+te_k-y)\ dy - \int_{\mathbb{R}^n} f(y)\phi_a(x-y)\ dy}{t} \\ &= \int_{\mathbb{R}^n} f(y)\left(\frac{\phi_a(x+te_k-y)-\phi_a(x-y)}{t}\right) dy. \end{aligned}$$

Since

$$\frac{\phi_a(x+te_k-y)-\phi_a(x-y)}{t} = \frac{\partial^k}{\partial x^k}\phi_a(x'-y)\bigg|_{x'=x+t'e_k,\,0\le t'\le t}$$

is less than some constant C in absolute value, the integrand above is dominated by $C\,|f| \in L^1$. Thus,

$$\begin{aligned} \frac{\partial g}{\partial x_k} &= \lim_{t\to 0}\frac{g(x+te_k)-g(x)}{t} \\ &= \int_{\mathbb{R}^n} f(y)\lim_{t\to 0}\left(\frac{\phi_a(x+te_k-y)-\phi_a(x-y)}{t}\right) dy \\ &= \int_{\mathbb{R}^n} f(y)\frac{\partial}{\partial x_k}\phi_a(x-y)\ dy. \end{aligned}$$

Thus, the partial derivatives exist, and

$$\frac{\partial}{\partial x_k}\phi_a(x-y) \in C^0,$$

so by the first lemma, $\partial g/\partial x_k$ is also continuous. By induction, we can conclude that $g(x) \in C^\infty$. □

Lemma 0.3. *If $f \in C_c(\mathbb{R}^n)$, then $g \in C_c(\mathbb{R}^n)$.*

Proof. There exists $R > 0$ such that $f = 0$ on $B(0,R)^c$. Choose x so that $g(x) \neq 0$. Then there exists y such that $f(y)\phi_a(x-y) \neq 0$. If $f(y) \neq 0$, then $y \in B(0,R)$. If $\phi_a(x-y) \neq 0$, then $x - y \in B(0,a)$. Thus,

$$|x| = |x - y + y| \leq |x - y| + |y| \leq a + R,$$

and so $g(x) = 0$ if $|x| \leq R + a$. In other words, $g \in C_c^\infty(\mathbb{R}^n)$. □

Theorem 0.4. *C_c^∞ is dense in L^p.*

Proof for L^1. We proved previously that C_c is dense in L^1; we just need to prove that C_c^∞ is dense in C_c. Given $f \in C_c$, there exists $r > 0$ such that $f = 0$ on $B(0,r)^c$. Given $\epsilon > 0$, since $f \in C_c$, f is uniformly continuous means that there exists $a > 0$ such that $|x - y| \leq a \Rightarrow$

$$|f(x) - f(y)| \leq \frac{\epsilon}{\lambda(B(0, r+1))},$$

and we may make $0 < a \leq 1$.

Consider ϕ_a:

$$\int \phi_a \; dx = 1 \quad \text{and} \quad \int \phi_a(x-y) dy = 1.$$

Thus,

$$\begin{aligned}
|f * \phi_a(x) - f(x)| &= \left| \int (f(y) - f(x))\phi_a(x-y) \; dy \right| \\
&\leq \int |f(y) - f(x)| \, \phi_a(x-y) \; dy \\
&= \int_{|x-y| \leq a} |f(y) - f(x)| \, \phi_a(x-y) \; dy \\
&\leq \frac{\epsilon}{\lambda(b(0, r+1))} \int_{|x-y|\leq a} \phi_a(x-y) \; dy \\
&= \frac{\epsilon}{\lambda(B(0, r+1))}.
\end{aligned}$$

So we have that

$$\begin{aligned}
\|f * \phi_a - f\|_1 &= \int_{\mathbb{R}^n} |f * \phi_a(x) - f(x)| \; dx \\
&= \int_{B(0,r+1)} |f * \phi_a(x) - f(x)| \; dx \\
&\leq \frac{\epsilon}{\lambda(B(0, r+1)} \lambda(B(0, r+1)) = \epsilon.
\end{aligned}$$

□

In fact, more is true. We first need a lemma.

Lemma 0.5. *If $f \in L^1(\mathbb{R}^n)$, then*

$$\lim_{y \to 0} \int_{\mathbb{R}^n} |f(x+y) - f(x)| \; dx = 0.$$

Theorem 0.6. *Let $f \in L^p(\mathbb{R}^n)$ with $1 \leq p < \infty$. Then*

$$\lim_{a \to 0} \|f * \phi_a - f\|_p = 0.$$

Proof for L^1. We have

$$\begin{aligned}
\int (f * \phi_a(x) - f(x)) &= \int \left(\int (f(x-y) - f(x))\phi_a(y) \; dy \right) \; dx \\
&= \int \left(\int (f(x-y) - f(x))\phi_a(y) \; dx \right) \; dy \\
&= \int \phi_a(y) \int (f(x-y) - f(x)) \; dx \; dy \\
&\leq \int_{B(0,r)} \phi_a(y) \cdot \epsilon + \int_{B(0,r)} 2 \|f\|_1 \phi_a(y) \\
&\leq \epsilon + 2 \|f\|_1 \int_{B(0,r)} \phi_a(y) \\
&\to 0 \quad \text{for } a \text{ sufficiently small.}
\end{aligned}$$

□