# MEASURE AND INTEGRATION: LECTURE 20

## Convolutions

**Definition.** If $f$ and $g$ are measurable functions on $\mathbb{R}^n$, then the *convolution* of $f$ and $g$, denoted $f * g$, is defined formally as

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y)g(x-y)\ dy.$$

The operation is commutative and associative:

$$(f * g)(x) = (g * f)(x) \quad \text{and} \quad (f * g) * h = f * (g * h).$$

**Inequalities.** Let $f$ be a Lebesgue measurable function on $\mathbb{R}^n$. Then the function $f(x)$ considered as a function of $(x, x)$ in $\mathbb{R}^{2n}$ is Lebesgue measurable since $\mathcal{L}_n \times \mathcal{L}_n \subset \mathcal{L}_{2n}$. The linear transformation given by $(x, y) \mapsto (x-y, y)$ is invertible, and so $f(x-y)$ is a Lebesgue measurable function of $(x, y) \in \mathbb{R}^{2n}$. Thus, we see that $f(y)g(x-y)$ is measurable on $\mathbb{R}^{2n}$.

The next theorem asserts that if $f$ and $g$ are in $L^1(\mathbb{R}^n)$, then $f * g$ exists a.e. and $f * g \in L^1(\mathbb{R}^n)$. Since the product of two integrable functions need not be integrable, it is not obvious that $f * g$ exists a.e.

**Theorem 0.1.** *Assume $f, g \in L^1(\mathbb{R}^n)$. Then for a.e. $x \in \mathbb{R}^n$, the convolution $(f * g)(x)$ exists, $f * g \in L^1(\mathbb{R}^n)$, and*

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1 .$$

*Proof.* Assume that $f$ and $g$ are non-negative. Then $f(y)g(x-y)$ is a non-negative measurable function, and Fubini I implies

$$\int dx \int f(y)g(x-y)\ dy = \int dy \int f(y)g(x-y)\ dx.$$

The LHS equals $\int (f * g)(x)\ dx$, and the RHS is

$$\int f(y)\ dy \int g(x-y)\ dx = \int f(y)\ dy \cdot \int g(x)\ dx.$$

Thus $\|f * g\|_1 = \|f\|_1 \|g\|_1$. When $f$ and $g$ are not necessarily non-negative, we see that $|f| * |g|$ exists a.e. $\Rightarrow |f(y)g(x-y)|$ integable $\Rightarrow f(y)g(x-y)$ integrable $\Rightarrow f * g$ exists a.e. Since $|f * g| \leq |f| * |g|$, the theorem follows. □

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**Young's theorem.** Our next theorem generalizes the previous one.

**Theorem 0.2.** *Let $p, q, r \in [1, \infty]$ such that*

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1.$$

*If $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, then $f * g$ exists a.e. and $f * g \in L^r(\mathbb{R}^n)$. Moreover,*

$$\|f * g\|_r \leq \|f\|_p \|g\|_q .$$

*Proof.* Without loss of generality, let $\|f\|_p = \|g\|_p = 1$. The general case follows from the non-negative case, so assume $f, g \geq 0$. Applying Hölder's inequality,

$$\begin{aligned}(f * g)(x) &= \int \left(f(y)^{p/r} g(x-y)^{q/r}\right) f(y)^{1-p/r} g(x-y)^{1-q/r} \, dy \\ &\leq \left(\int f(y)^p g(x-y)^q \, dy\right)^{1/r} \left(\int f(y)^{(1-p/r)q'} \, dy\right)^{1/q'} \\ &\quad \times \left(\int g(x-y)^{(1-q/r)p'} \, dy\right)^{1/p'} .\end{aligned}$$

We have used the fact that

$$\frac{1}{r} + \frac{1}{q'} + \frac{1}{p'} = \frac{1}{r} + \left(1 - \frac{1}{q}\right) + \left(1 - \frac{1}{p}\right) = 1.$$

Since

$$\begin{aligned}\left(1 - \frac{p}{r}\right) q' &= p \left(\frac{1}{p} - \frac{1}{r}\right) q' = p \left(1 - \frac{1}{q}\right) = p, \\ \left(1 - \frac{q}{r}\right) p' &= q \left(\frac{1}{q} - \frac{1}{r}\right) p' = q \left(1 - \frac{1}{p}\right) p' = q,\end{aligned}$$

we have

$$(f * g)(x) \leq \left(\int f(y)^p g(x-y)^q \, dy\right)^{1/r} \cdot 1 \cdot 1,$$

i.e.,

$$(f * g)^r(x) \leq \int f(y)^p g(x-y)^q \, dy.$$

Thus, $(f * g)^r \leq f^p * g^q$, and so

$$\begin{aligned}\int (f * g)\ dx &\leq \|f^p * g^q\|_1 \\ &= \|f^p\|_1 \|g^q\|_1 \\ &= \|f\|_p^p \|g\|_q^q \\ &= 1.\end{aligned}$$

□

The proof ignores the case in which some of the exponents equal $\infty$. But, if $p = \infty$, then $r = \infty$ and $q = 1$, and the result follows since $|f * g| \leq \|f\|_\infty \|g\|_1$. If $r = \infty$, then $q = p'$, and the result follows from Hölder's inequality. However, more is true when $r = \infty$.

**Theorem 0.3.** *Let $1 \leq p \leq \infty$ and $f \in L^p(\mathbb{R}^n)$. Then the integral defining $(f * g)(x)$ exists for all $x \in \mathbb{R}^n$, $f * g$ is bounded and uniformly continuous, and if $1 < p < \infty$, then $f * g \in C_0$ (i.e., $\lim_{|x|\to\infty}(f * g)(x) = 0$).*

*Proof.* Either $p$ or $p'$ must be finite. Suppose $p' < \infty$. The corollary to $C_c$ dense in $L^p$ implies that for all $\epsilon > 0$ there exists $\delta > 0$ such that if $|y| < \delta$, then $\|\tau_y g - g\|_{p'} \leq \epsilon$, where $\tau$ is translation by $y$. Thus, $|x - x'| \leq \delta$, then

$$\|\tau_x g - \tau_{x'} g\|_{p'} = \|\tau_{x-x'} g - g\|_{p'} \leq \epsilon.$$

By Hölder's inequality,

$$\begin{aligned}|(f * g)(x) - (f * g)(x')| &\leq \int |f(y)|\, |g(x - y) - g(x' - y|\ dy \\ &= \int |f(-y)|\, |g(x + y) - g(x' + y)|\ dy \\ &\leq \|f\|_p \|\tau_x g - \tau_{x'} g\|_{p'} \\ &\leq \|f\|_p \epsilon.\end{aligned}$$

This proves that $f * g$ is uniformly continuous.

Now let $1 < p < \infty$. Since $C_c$ is dense in $L^p$, there exist sequences $f_k, g_k \in C_c(\mathbb{R}^n)$ such that $f_k \to f$ in $L^p$ and $g_k \to g$ in $L^{p'}$. Thus, $f_k * g_k \in C_c(\mathbb{R}^n)$. Estimating,

$$\begin{aligned}\|f_k * g_k - f * g\|_\infty &\leq \|f_k * (g_k - g)\|_\infty + \|(f_k - f) * g\|_\infty \\ &\leq \|f_k\|_p \|g_k - g\|_{p'} + \|f_k - f\|_p \|g\|_{p6\prime} \\ &\to 0 \quad \text{as } k \to \infty.\end{aligned}$$

Thus, $f_k * g_k$ converges uniformly to $f * g$, and so $f * g \to 0$ as $|x| \to \infty$. □