MEASURE AND INTEGRATION: LECTURE 18

Fubini's theorem

Notation. Let ℓ and m be positive integers, and $n = \ell + m$. Write $\mathbb{R}^n$ as the Cartesian product $\mathbb{R}^n = \mathbb{R}^\ell + \mathbb{R}^m$. We will write points in $\mathbb{R}^n$ as

$$z \in \mathbb{R}^n; \quad x \in \mathbb{R}^\ell; \quad y \in \mathbb{R}^m;$$
$$z = (x, y).$$

If f is a function on $\mathbb{R}^n$ and $y \in \mathbb{R}^m$ is fixed, then f_y is the function on $\mathbb{R}^\ell$ defined by

$$f_y(x) = f(x, y).$$

The function f_y is called the *section* of f determined by y. In particular, if $A \subset \mathbb{R}^n$ and $f = \chi_A$, then

$$f_y(x) = \begin{cases} 1 & \text{if } (x, y) \in A; \\ 0 & \text{if } (x, y) \notin A. \end{cases}$$

In this case, f_y is the characteristic function of a subset of $\mathbb{R}^\ell$, and a point $x \in \mathbb{R}^\ell$ is in this set if and only if $(x, y) \in A$. This set will be denoted by

$$A_y = \{x \in \mathbb{R}^\ell \mid (x, y) \in A\},$$

and is called the *section* of A determined by y.

Now let f be any function on $\mathbb{R}^n$. For a fixed $y \in \mathbb{R}^m$, it may be that the function f_y on $\mathbb{R}^\ell$ is integrable. In this case, let

$$F(y) = \int_{\mathbb{R}^\ell} f_y(x)\, dx.$$

Of course, f_y must be $\mathcal{L}$-measurable, but there are two ways $F(y)$ could exist: (1) $f_y \geq 0$, in which case $0 \leq F(y) \leq \infty$, and (2) $f_y \in L^1(\mathbb{R}^\ell)$, in which case $-\infty < F(y) < \infty$.

We eventually want to prove the equation

$$\int_{\mathbb{R}^m} F(y)\, dy = \int_{\mathbb{R}^n} f(z)\, dz.$$

To show this, we assume f is $\mathcal{L}$-measurable and integrable, and prove that $F(y)$ exists for a.e. $y \in \mathbb{R}^m$ and that F is $\mathcal{L}$-measurable and integrable on $\mathbb{R}^m$.

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However, it cannot be expected that f_y is an $\mathcal{L}$-measurable function for all $y \in \mathbb{R}^m$. Indeed, let $E \subset \mathbb{R}^\ell$ be a non-measurable set, fix $y_0 \in \mathbb{R}^m$, and let $A = E \times \{y_0\}$. Then $A_y = \emptyset$ if $y \neq y_0$ but $A_{y_0} = E$. The set A is measurable with $\lambda(A) = 0$. But A_{y_0} is not measurable.

Fubini I: Non-negative functions.

Theorem 0.1. *Assume that $f\colon \mathbb{R}^n \to [0,\infty]$ is $\mathcal{L}$-measurable. Then for a.e. $y \in \mathbb{R}^m$, the function $f_y\colon \mathbb{R}^\ell \to [0,\infty]$ is $\mathcal{L}$-measurable, and so*

$$F(y) = \int_{\mathbb{R}^\ell} f_y(x)\ dx$$

exists. Moreover, F is $\mathcal{L}$-measurable on $\mathbb{R}^m$, and

$$\int_{\mathbb{R}^m} F(y)\ dy = \int_{\mathbb{R}^n} f(x)\ dz.$$

The second equation will be abbreviated

$$\int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^\ell} f(x,y)\ dx \right)\ dy = \int_{\mathbb{R}^n} f(x,y)\ dx\ dy,$$

and the LHS of this equation will often be written

$$\int_{\mathbb{R}^m} \int_{\mathbb{R}^\ell} f(x,y)\ dx\ dy \quad \text{or} \quad \int_{\mathbb{R}^m} dy \int_{\mathbb{R}^\ell} f(x,y)\ dx.$$

Proof. The proof is long, and is broken into 10 steps.

(1) Let J be a special rectangle. Then $J = J_1 \times J_2$, with J_1 and J_2 special rectangles in $\mathbb{R}^\ell$ and $\mathbb{R}^m$. Then for any $y \in \mathbb{R}^m$,

$$J_y = \begin{cases} J_1 & \text{if } y \in J_2; \\ \emptyset & \text{if } y \notin J_2. \end{cases}$$

Thus, $\lambda(J_y) = \lambda(J_1)\chi_{J_2}(y)$, and so

$$\begin{aligned} \int_{\mathbb{R}^m} \lambda(J_y)\ dy &= \lambda(J_1)\lambda(J_2) \\ &= \lambda(J). \end{aligned}$$

(2) Let $G \subset \mathbb{R}^n$ be open, and write $G = \cup_{k=1}^\infty J_k$, with each J_k a disjoint rectangle. Thus,

$$G_y = \bigcup_{k=1}^\infty J_{k,y}$$

is a disjoint union, and so $\lambda(G_y) = \sum \lambda(J_{k,y})$. Thus,

$$\begin{aligned}\int_{\mathbb{R}^m} \lambda(G_y)\ dy &= \sum_{k=1}^{\infty} \int_{\mathbb{R}^m} \lambda(J_{k,y})\ dy \\ &= \sum_{k=1}^{\infty} \lambda(J_k) \\ &= \lambda(G).\end{aligned}$$

(3) Let $K \subset \mathbb{R}^n$ be compact, and choose $G \supset K$ open and bounded. Apply (2) to $G \setminus K$:

$$\int_{\mathbb{R}^m} \lambda(G_y \setminus K_y)\ dy = \lambda(G \setminus K);$$

$$\int_{\mathbb{R}^m} \lambda(G_y)\ dy - \int_{\mathbb{R}^m} \lambda(K_y)\ dy = \lambda(G) - \lambda(K).$$

Thus, applying (2) to G gives

$$\int_{\mathbb{R}^m} \lambda(K_y)\ dy = \lambda(K).$$

(4) Let $K_1 \subset K_2 \subset \cdots$ be compact. Let $B = \cup_k K_j$. Then for all $y \in \mathbb{R}^m$,

$$B_y = \bigcup_{j=1}^{\infty} K_{j,y}.$$

So B_y is measurable, $\lambda(B_y) = \lim_j \to \infty \lambda(K_{j,y})$ is increasing, so by monotone convergence,

$$\begin{aligned}\int_{\mathbb{R}^m} \lambda(B_y)\ dy &= \lim_{j\to\infty} \int_{\mathbb{R}^m} \lambda(K_{j,y})\ dy \\ &= \lim_{j\to\infty} \lambda(K_j) \quad \text{by (3)} \\ &= \lambda(B).\end{aligned}$$

(5) Let $G_1 \supset G_2 \supset \cdots$ be open and bounded. Let $C = \cap_j G_j$ and let $K \supset G_1$. Applying (4),

$$K \setminus C = \bigcup_{j=1}^{\infty} (K \setminus G_j),$$

and so

$$\int_{\mathbb{R}^m} \lambda(K_y \setminus C_y)\ dy = \lambda(K \setminus C).$$

Since

$$\int_{\mathbb{R}^m} \lambda(K_y)\ dy = \lambda(K),$$

the result follows for C.

(6) This step is the most important. Let A be bounded and measurable. By the approximation theorem, there exist compact sets K_j and bounded open sets G_j such that

$$K_1 \subset K_2 \subset \cdots \subset A \subset \cdots \subset G_2 \subset G_1$$

and

$$\lim_{j\to\infty} \lambda(K_j) = \lambda(A) = \lim_{j\to\infty} \lambda(G_j).$$

Let

$$B = \bigcup_{j=1}^{\infty} K_j \quad \text{and} \quad C = \bigcap_{j=1}^{\infty}.$$

Then $B \subset A \subset C$ and $\lambda(B) = \lambda(A) = \lambda(C)$. Thus, by (4) and (5),

$$\int_{\mathbb{R}^m} (\lambda(C_y) - \lambda(B_y)) \ dy = 0,$$

and so $\lambda(C_y) - \lambda(B_y) = 0$ for a.e. $y \in \mathbb{R}^m$. This means that $C_y \setminus B_y$ has measure zero in $\mathbb{R}^\ell$ for a.e. $y \in \mathbb{R}^m$, and for these y, $B_y \subset A_y \subset C_y \Rightarrow A_y = B_y \cup N$, where N is a null set. Hence, A_y is measurable for a.e. y, $\lambda(A_y)$ is a measurable function of y, and

$$\begin{aligned}\int_{\mathbb{R}^m} \lambda(A_y) \ dy &= \int_{\mathbb{R}^m} \lambda(B_y) \ dy \\ &= \lambda(B) = \lambda(A).\end{aligned}$$

(7) Observe that if the theorem is valid for each function $0 \le f_1 \le f_2 \le \cdots$, then it is valid for $f = \lim f_j$. This is due to monotone convergence, (2) and (4). Since $f_{j,y}$ is measurable for a.e. y, f_y is $\mathcal{L}$-measurable for a.e. y, and thus for a.e. $y \in \mathbb{R}^m$,

$$\begin{aligned}F(y) &= \int_{\mathbb{R}^\ell} f_y(x) \ dx \\ &= \lim_{j\to\infty} \int_{\mathbb{R}^\ell} f_{j,y}(x) \ dx \\ &= \lim_{j\to\infty} F_j(y).\end{aligned}$$

Since this is an increasing limit and F_j is measurable, so is F, and by monotone convergence,

$$\begin{aligned}\int_{\mathbb{R}^m} F(y)\ dy &= \lim_{j\to\infty}\int_{\mathbb{R}^m} F_j(y)\ dy\\ &= \lim_{j\to\infty}\int_{\mathbb{R}^n} f_j(z)\ dz\\ &= \int_{\mathbb{R}^n} f(z)\ dz.\end{aligned}$$

(8) Let f_j be the characteristic function of the bounded set $A \cap B(0, j)$. Then the theorem is valid for the characteristic function of any measurable set by (6) the observation in (7).

(9) Since non-negative measurable simple functions are (finite) linear combinations of functions in (8), the theorem follows for them.

(10) The theorem follows from the theorem that states that there exists a sequence of simple measurable functions converging to any non-negative measurable function.

□

Fubini II: Integrable functions.

Theorem 0.2. *Assume that* $f \in L^1(\mathbb{R}^n)$. *Then for a.e.* $y \in \mathbb{R}^m$, *the function* $f_y \in L^1(\mathbb{R}^\ell)$, *and*

$$F(y) = \int_{\mathbb{R}^\ell} f_y(x)\ dx$$

exists. Moreover, $F \in L^1(\mathbb{R}^m)$, *and*

$$\int_{\mathbb{R}^m} F(y)\ dy = \int_{\mathbb{R}^n} f(z)\ dz.$$

Proof. Write $f = f^+ - f^-$ and apply Fubini I. Define

$$G(y) = \int_{\mathbb{R}^\ell} f_y^-\ dx, \quad H(y) = \int_{\mathbb{R}^\ell} f_y^+\ dx,$$

so that

$$\int_{\mathbb{R}^m} G\ dy = \int_{\mathbb{R}^n} f^-\ dz, \quad \int_{\mathbb{R}^m} H\ dy = \int_{\mathbb{R}^n} f^+\ dz.$$

Because the integrals are finite, $G(y) < \infty$ and $H(y) < \infty$ a.e. and thus $f_y \in L^1(\mathbb{R}^\ell)$. Also, $F(y) = H(y) - G(y)$ a.e., and so $F \in L^1(\mathbb{R}^m)$

and

$$\begin{aligned}\int_{\mathbb{R}^m} F\ dy &= \int_{\mathbb{R}^m} H\ dy - \int_{\mathbb{R}^m} G\ dy \\ &= \int_{\mathbb{R}^n} f^+\ dz - \int_{\mathbb{R}^n} f^-\ dz \\ &= \int_{\mathbb{R}^n} f\ dz.\end{aligned}$$

□

Example of Fubini's theorem. Let us calculate the integral

$$\int_E y \sin x e^{-xy}\ dx\ dy,$$

where $E = (0,\infty) \times (0,1)$. Since the integrand is a a continuous function, it is $\mathcal{L}$-measurable. We have by integration by parts

$$\begin{aligned}F(y) &= \int_0^\infty y \sin x e^{-xy}\ dx \\ &= \frac{y}{y^2+1}.\end{aligned}$$

Thus,

$$\int_0^1 F(y)\ dy = \frac{1}{2}\log 2.$$

Now, since $|f(x,y)| \le ye^{-xy}$, we may apply Fubini I to see that

$$\begin{aligned}\int_E |f(x,y)|\ dx\ dy &\le \int_E ye^{-xy}\ dx\ dy \\ &= \int_0^1\ dy \int_0^\infty ye^{-xy}\ dx \\ &= \int_0^1\ dy \\ &= 1.\end{aligned}$$

Doing integration with respect to y first yields

$$\int_0^1 y \sin x e^{-xy}\ dy = \frac{\sin x}{x}\left(\frac{1-e^{-x}}{x} - e^{-x}\right).$$

Thus, Fubini's theorem shows that

$$\int_0^\infty \frac{\sin x}{x}\left(\frac{1-e^{-x}}{x} - e^{-x}\right)\ dx = \frac{1}{2}\log 2.$$