MEASURE AND INTEGRATION: LECTURE 17

Inclusions between L^p spaces. Consider Lebesgue measure on the space $(0,\infty) \subset \mathbb{R}$. Recall that x^a is integrable on $(0,1) \iff a > -1$, and it is integrable on $(1,\infty) \iff a < -1$. Now let $1 \le p < q \le \infty$. Choose b such that $1/q < b < 1/p$. Then $x^{-b}\chi_{(0,1)}$ is in L^p but not in L^q, which shows that $L^p \not\subset L^q$. On the other hand, $x^{-b}\chi_{(1,\infty)}$ is in L^q but not in L^p, so that $L^q \not\subset L^p$. Thus, in general there is no inclusion relation between two L^p spaces.

The limit of $\|f\|_p$ as $p \to \infty$. For convenience, define $\|f\|_p$ to be ∞ if f is $\mathcal{M}$-measurable but $f \notin L^p$.

Theorem 0.1. *Let $f \in L^r$ for some $r < \infty$. Then*

$$\lim_{p\to\infty} \|f\|_p = \|f\|_\infty .$$

This justifies the notation for the L^∞ norm.

Proof. Let $t \in [0, \|f\|_\infty)$. By definition, the set

$$A = \{x \in X \mid |f(x)| \ge t\}$$

has positive measure. Observe the trivial inequality

$$\begin{aligned} \|f\|_p &\ge \left(\int_A |f|^p \, d\mu\right)^{1/p} \\ &\ge (t^p \mu(A))^{1/p} \\ &= t\mu(A)^{1/p}. \end{aligned}$$

If $\mu(A)$ is finite, then $\mu(A)^{1/p} \to 1$ as $p \to \infty$. If $\mu(A) = \infty$, then $\mu(A)^{1/p} = \infty$. In both cases, we have

$$\liminf_{p\to\infty} \|f\|_p \ge t.$$

Since t is arbitrary,

$$\liminf_{p\to\infty} \|f\|_p \ge \|f\|_\infty .$$

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For the reverse inequality, we need the assumption that $f \in L^r$ for some (finite) r. For $r < p < \infty$, we have

$$\|f\|_p \leq \|f\|_r^{r/p} \|f\|_\infty^{1-r/p}.$$

Since $\|f\|_r < \infty$,

$$\limsup_{p\to\infty} \|f\|_p \leq \|f\|_\infty.$$

□

The inequality used in the proof can be written as

$$\mu(\{x \in X \mid |f(x)| \geq t\}) \leq \left(\frac{\|f\|_p}{t}\right)^p,$$

and is known as *Chebyshev's inequality*.

Finite measure spaces. If the measure of the space X is finite, then there are inclusion relations between L^p spaces. To exclude trivialities, we will assume throughout that $0 < \mu(X) < \infty$.

Theorem 0.2. *If $q \leq p < q < \infty$, then $L^q \subset L^p$.*

Proof. Applying Hölder's inequality to $|f|^p$ and 1,

$$\begin{aligned}\int |f|^p \ d\mu &= \int |f|^p \cdot 1 \ d\mu \\ &\leq \left(\int |f|^{pq/p} \ d\mu\right)^{p/q} \left(\int \ d\mu\right)^{1-p/q} \\ &= \left(\int |f|^q \ d\mu\right)^{p/q} \mu(X)^{1-p/q}.\end{aligned}$$

□

In particular, if $\mu(X) = 1$, then

$$\|f\|_1 \leq \|f\|_p \leq \|f\|_q \leq \|f\|_\infty.$$

Counting measure and l^p spaces. Let X be any set, $\mathcal{M} = \mathcal{P}(X)$, and μ be the counting measure. Recall that $\mu(A)$ is the number of points in A if A is finite and equals ∞ otherwise. Integration is simply

$$\int_X f \ d\mu = \sum_{x\in X} f(x)$$

for any non-negative function f, and L^p is denoted by l^p.

Theorem 0.3. *If $1 \leq p < q \leq \infty$, then $l^p \subset l^q$, and*

$$\|f\|_\infty \leq \|f\|_q \leq \|f\|_p \leq \|f\|_1.$$

Proof. If $q = \infty$, then observe that for any $x_0 \in X$,

$$|f(x_0)| \leq \left(\sum_{x \in X} |f(x)|^p \right)^{1/p}.$$

Now let $q < \infty$. Then we NTS

$$\left(\sum_{x \in X} |f(x)|^q \right)^{1/q} \leq \left(\sum_{x \in X} |f(x)|^p \right)^{1/p}.$$

Now multiply both sides by a constant so that the RHS is equal to 1. Thus, assuming $\sum |f(x)|^p = 1$, we NTS that $\sum |f(x)|^q \leq 1$. But this is immediate, since $|f(x)| \leq 1$ for all x implies that $|f(x)|^q \leq |f(x)|^p$ because $q > p$. □

Thus, in a certain sense, the counting measure and a finite measure act in reverse ways for L^p spaces.

Local L^P spaces. Let G be an open set in $\mathbb{R}^n$. The *local L^p space* on G consists of all $\mathcal{L}$-measurable functions f defined a.e. on G such that for every compact set $K \subset G$, the characteristic function $f\chi_K$ has a finite L^p norm; that is,

$$\int_K |f(x)|^p \; dx < \infty \quad \text{if } 1 \leq p < \infty;$$
$$f \text{ is essentially bounded on } K \quad \text{if } p = \infty.$$

This set is denoted $L^p_{\text{loc}}(G)$. From our result on finite measure spaces, we have at once for $1 \leq p < q \leq \infty$,

$$L^\infty_{\text{loc}}(G) \subset L^q_{\text{loc}}(G) \subset L^p_{\text{loc}}(G) \subset L^1_{\text{loc}}(G).$$

Convexity properties of L^p norm. Let $(X, \mathcal{M}, \mu)$ be a measure space.

Theorem 0.4. *Let $1 \leq p < r < q < \infty$ and suppose $f \in L^p \cap L^q$. Then $f \in L^r$ and*

$$\log \|f\|_r \leq \frac{\frac{1}{r} - \frac{1}{q}}{\frac{1}{p} - \frac{1}{q}} \log \|f\|_p + \frac{\frac{1}{p} - \frac{1}{r}}{\frac{1}{p} - \frac{1}{q}} \log \|f\|_q .$$

Proof. Since $1/q < 1/r < 1/p$, there exists a unique θ such that

$$\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}.$$

The number θ satisfies $0 < \theta < 1$ and equals

$$\theta = \frac{\frac{1}{r} - \frac{1}{q}}{\frac{1}{p} - \frac{1}{q}}, \quad 1 - \theta = \frac{\frac{1}{p} - \frac{1}{r}}{\frac{1}{p} - \frac{1}{q}}.$$

We NTS that $\log \|f\|_r \leq \theta \log \|f\|_p + (1-\theta) \log \|f\|_q$. Note that

$$1 = \frac{r\theta}{p} + \frac{r(1-\theta)}{q},$$

and so $p/r\theta$ and $q/r(1-\theta)$ are conjugate exponents. Thus, by Hölder's inequality,

$$\begin{aligned} \|f\|_r &= \left\|f^\theta f^{1-\theta}\right\|_r \\ &= \left\|f^{r\theta} f^{r(1-\theta)}\right\|_1^{1/r} \\ &\leq \left(\left\|f^{r\theta}\right\|_{p/r\theta} \left\|f^{r(1-\theta)}\right\|_{q/r(1-\theta)}\right)^{1/r} \\ &= \left(\|f\|_p^{r\theta} \|f\|_q^{r(1-\theta)}\right)^{1/r} \\ &= \|f\|_p^\theta \|f\|_q^{1-\theta}. \end{aligned}$$

□

The theorem states that if f is an $\mathcal{M}$-measurable non-zero function on X, then the set of indices p such that $f \in L^p$ is an interval $I \subset [1, \infty]$, and $\log \|f\|_p$ is a convex function of $1/p$ on I.