MEASURE AND INTEGRATION: LECTURE 16

C_c dense in L^p for $1 \leq p < \infty$.

Theorem 0.1. *Let*

$$S = \{s \colon X \to \mathbb{C} \mid s \text{ simple, measurable such that } \mu(\{x \mid s(x) \neq 0\})\}.$$

For $1 \leq p < \infty$, S is dense in $L^p(\mu)$, i.e., given $f \in L^p(\mu)$ there exists sequence $s_k \in S$ such that $\|s_k - f\|_p \to 0$.

Proof. Note that $S \subset L^p(\mu)$ since

$$\int_X s^p \, d\mu \leq \max s^p \mu(\{x \mid f(x) \neq 0\}) < \infty.$$

If $f \colon X \to \mathbb{R}$ and $f \geq 0$, then by the approximation theorem, there exists s_k simple measurable functions such that $0 \leq s_1 \leq \cdots \leq f$ and $\lim_{k\to\infty} s_k = f$. Since $s_k \leq f$, $\int s_k^p \leq \int f^p < \infty$. Thus,

$$s_k \in L^p \Rightarrow s_k \in S.$$

We have

$$f - s_n \leq \Rightarrow |f - s_n|^p \leq |f|^p.$$

So $|f - s_n|^p \leq f = |f|^p \in L^1$, and we can apply the dominated convergence theorem. Thus,

$$\lim \int_X |f - s_n|^p \, d\mu = \int_X \lim (f - s_n)^p \, d\mu - 0,$$

and so $\|s_n - f\|_p \to 0$. If f is not non-negative, apply separately to f^+ and f^-. □

Corollary 0.2. *If X is a locally compact Hausdorff space, then for $1 \leq p < \infty$, $C_c(X)$ is dense in L^p.*

Proof. Let S be as in the previous theorem. If $s \in S$ and $\epsilon > 0$, there exists $g \in C_c(X)$ such that $\mu(\{x \mid g(x) \neq s(x)\}) < \epsilon$ by Lusin's

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theorem, and also $|g| \leq \|s\|_\infty$. Thus,

$$\begin{aligned}\|g-s\|_p &= \left(\int |g-s|^p\right)^{1/p} = \left(\int_{g=s} |g-s|^p + \int_{g\neq s} |g-s|^p\right)^{1/p} \\ &= \left(\int_{g\neq s} |g-s|^p\right)^{1/p} \leq \left(\int_{g\neq s} 2^p \|s\|_\infty^p\right)^{1/p} \\ &\leq 2\|s\|_\infty \epsilon^{1/p}.\end{aligned}$$

Thus, $C_c(X)$ is dense in S, and since S is dense in L^p, $C_c(X)$ is dense in L^p. □

Here is an example. Let $X = \mathbb{R}^n$ and let $f, g \in C_c(\mathbb{R}^n)$. Define $d(f,g) = \int_{-\infty}^{\infty} |f(t) - g(t)|\ dt$. Note that $C_c(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$ and L^1 is complete. The space $L^1(\mathbb{R}^n)$ is the completion of $C_c(\mathbb{R}^n)$ under this metric, provided $f \sim g$ if $f = g$ a.e. Any metric space has a unique completion under its metric.

The case $p = \infty$. Let $f, g \in C_c(X)$ and

$$d(f,g) = \sup_{x\in X} |f(x) - g(x)|.$$

Then L^∞ is not the completion of $C_c(X)$ under d.

A function $f\colon X \to \mathbb{C}$ *vanishes at infinity* if for every $\epsilon > 0$ there exists a compact subset $K \subset X$ such that $|f(x)| < \epsilon$ whenever $x \notin K$. The set of all continuous function that vanish at infinity is denoted by $C_0(x)$.

C_c dense in C_0.

Theorem 0.3. *The completion of $C_c(X)$ under $\|\cdot\|_\infty$ is $C_0(X)$.*

Proof. We show that (a) $C_c(X)$ is dense in $C_0(X)$, and (b) $C_0(X)$ is complete.

Proof of (a). Let $f \in C_0(X)$. For $\epsilon > 0$, there exists K compact such that $|f(x)| < \epsilon$ for all $x \in K^c$. By Urysohn's lemma, there exists $g \in C_c(X)$ such that $K \prec g$, $0 \leq g \leq 1$, and $g = 1$ on K. Let $h = fg$. Then $h \in C_c(X)$ and $\|f - h\|_\infty < \epsilon$. ($f = h$ on K and $f < \epsilon$ on K^c.)

Proof of (b). Let f_n be a Cauchy sequence in $C_0(X)$, i.e., given $\epsilon > 0$, there exists N such that $i, j > N$, $\|f_i(x) - f_j(x)\|_\infty < \epsilon$. In other words, f_n converges uniformly. Thus,

$$\lim_{n\to\infty} f_n = f \text{ exists}$$

and f is continuous. Given $\epsilon > 0$, there exists n such that $\|f_n - f\|_\infty < \epsilon/2$ and there exists K compact such that $|f_n(x)| < \epsilon/2$ for all $x \in K^c$. Then $|f| = |f - f_n + f_n| \leq \epsilon/2 + \epsilon/2 = \epsilon$ on K^c. Thus, $f \in C_0(X)$. □