MEASURE AND INTEGRATION: LECTURE 15

L^p spaces. Let $0 < p < \infty$ and let $f\colon X \to \mathbb{C}$ be a measurable function. We define the L^p *norm* to be

$$\|f\|_p = \left(\int_X |f|^p \, d\mu \right)^{1/p},$$

and the space L^p to be

$$L^p(\mu) = \{f\colon X \to \mathbb{C} \mid f \text{ is measurable and } \|f\|_p < \infty\}.$$

Observe that $\|f\|_p = 0$ if and only if $f = 0$ a.e. Thus, if we make the equivalence relation $f \sim g \iff f = g$ a.e, then $\|\cdot\|$ makes L^p a normed space (we will define this later).

If μ is the counting measure on a countable set X, then

$$\int_X f \, d\mu = \sum_{x \in X} f(x).$$

Then L^p is usually denoted ℓ^p, the set of sequences s_n such that

$$\left(\sum_{n=1}^{\infty} |s_n|^p \right)^{1/p} < \infty.$$

A function f is *essentially bounded* if there exists $0 \leq M < \infty$ such that $|f(x)| \leq M$ for a.e. $x \in X$. The space L^∞ is defined as

$$L^\infty(\mu) = \{f\colon X \to \mathbb{C} \mid f \text{ essentially bounded}\}$$

with the L^∞ norm

$$\|f\|_\infty = \inf\{M \mid |f(x)| \leq M \text{ a.e. } x \in X\}.$$

Proposition 0.1. *If $f \in L^\infty$, then $|f(x)| \leq \|f\|_\infty$ a.e.*

Proof. By definition of inf, there exists $M_k \to \|f\|_\infty$ such that $|f(x)| < M_k$ a.e, or, equivalently, there exists N_k with $\mu(N_k) = 0$ such that $|f(x)| \leq M_k$ for all $x \in N_k^c$. Let $N = \cup_{k=1}^\infty N_k$. Then $\mu(N) = 0$. If $x \in N^c = \cap_{k=1}^\infty (N_k)^c$, then $|f(x)| \leq M_k$ since $M_k \to \|f\|_\infty$. Thus, $|f(x)| \leq \|f\|_\infty$ for all $x \in N^c$. $\square$

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Theorem 0.2. *Let $1 \le p \le \infty$ and $1/p + 1/q = 1$. Let $f \in L^p(\mu)$ and $g \in L^q(\mu)$. Then $fg \in L^1(\mu)$ and*

$$\|fg\|_1 \le \|f\|_p \|g\|_q \text{ i.e.,}$$

$$\int |fg| \; d\mu \le \left(\int |f|^p\right)^{1/p} \left(\int |g|^q\right)^{1/q}.$$

Proof. If $1 < p < \infty$, this is simply Hölder's inequality. If $p = 1$, $q = \infty$, then $|f(x)g(x)| \le \|g\|_\infty |f(x)|$ a.e. Thus,

$$\int |fg| \le \|g\| \int |f|.$$

□

Theorem 0.3. *Let $1 \le p \le \infty$. Let $f, g \in L^p(\mu)$. Then $f + g \in L^p(\mu)$ and $\|f + g\|_p \le \|f\|_p + \|g\|_p$.*

Proof. If $1 < p < \infty$, this is simply Minkowski's inequality. If $p = 1$, then $\int |f + g| \le \int |f| + \int |g|$ is true. If $p = \infty$, then $|f + g| \le |f| + |g| \Rightarrow \|f + g\|_\infty \le \|f\|_\infty + \|g\|_\infty$. □

Normed space and Banach spaces. A *normed space* is a vector space V together with a function $\|\cdot\| : V \to \mathbb{R}$ such that

(a) $0 \le \|x\| < \infty$.
(b) $\|x\| = 0 \iff x = 0$.
(c) $\|\alpha x\| = |\alpha| \, \|x\|$ for all $\alpha \in \mathbb{C}$.
(d) $\|x + y\| \le \|x\| + \|y\|$.

For example, $L^p(\mu)$ is a normed space if two functions f, g are considered equal if and only if $f = g$ a.e. Also, $\mathbb{R}^n$ with the Euclidean norm is a normed space.

A *metric space* is a set M together with a function $d \colon M \times M \to \mathbb{R}$ such that

(a) $0 \le d(x, y) < \infty$.
(b) $d(x, x) = 0$.
(c) $d(x, y) > 0$ if $x \ne y$.
(d) $d(x, y) = d(y, x)$.
(e) $d(x, y) \le d(x, z) + d(z, y)$.

A normed space is a metric space with metric $d(f, g) = \|f - g\|$.

Recall that $x_i \to x \in M$ if $\lim_{n\to\infty} d(x_n, x) = 0$. A sequence $\{x_i\}$ is Cauchy if for every $\epsilon > 0$ there exists $N(\epsilon)$ such that $d(x_j, x_k) \le \epsilon$ for all $j, k \ge N(\epsilon)$.

Claim: if $x_n \to x$, then it is Cauchy. We know that $\lim_{n\to\infty} d(x_n, x) = 0$, so given $\epsilon > 0$, there exists N such that $d(x_k, x) < \epsilon/2$ for all $k > N$. for $j, k > N$, $d(x_k, x_j) \le d(x_j, x) + d(x, x_k) < \epsilon$.

However, a Cauchy sequence does not have to converge. For example, consider the space $\mathbb{R} \setminus \{0\}$ (the punctured real line) with the absolute value norm. The sequence $x_n = 1/n$ is Cauchy but it does not converge to a point in the space.

A metric space is called *complete* if every Cauchy sequence converges. By the Bolzano-Weierstrass theorem, $\mathbb{R}^n$ is complete. (Every Cauchy sequence is bounded, so it has a convergent subsequence and must converge.)

A normed space $(V, \|\cdot\|)$ that is complete under the induced metric $d(f,g) = \|f - g\|$ is called a *Banach space*.

Riesz-Fischer theorem.

Lemma 0.4. *If $\{f_n\}$ is Cauchy, then there exists a subsequence f_{n_k} such that $d(f_{n_{k+1}}, f_{n_k}) \leq 2^{-k}$.*

Theorem 0.5. *For $1 \leq p \leq \infty$ and for any measure space $(X, \mathcal{M}, \mu)$, the space $L^p(\mu)$ is a Banach space.*

Proof. Let $1 \leq p < \infty$ and let $\{f_n\} \in L^p(\mu)$ be a Cauchy sequence. By the lemma, there exists a subsequence n_k with $n_1 < n_2 < \cdots$ such that $\left\|f_{n_{k+1}}, f_{n_k}\right\|_p < 2^{-k}$. Let $g_k = \sum_{i=1}^k \left\|f_{n_{i+1}} - f_{n_i}\right\|_p$ and $g = \lim_{k\to\infty} g_k = \sum_{i=1}^\infty \left\|f_{n_{i+1}} - f_{n_i}\right\|_p$. By Minkowski's inequality,

$$\|g_k\|_p \leq \sum_{i=1}^k \left\|f_{n_{i+1}} - f_{n_i}\right\|_p < \sum_{i=1}^k 2^{-i} < 1.$$

Consider g_k^p. By Fatou's lemma,

$$\int \liminf g_k^p \leq \liminf \int g_k^p,$$

and so

$$\int g^p \leq 1 \Rightarrow g(x) < \infty \text{ a.e.}$$

Thus, the series

$$f_{n_1}(x) + \sum_{i=1}^\infty (f_{n_{i+1}}(x) - f_{n_i}(x))$$

converges absolutely a.e. Define

$$f(x) = \begin{cases} f_{n_1}(x) + \sum_{i=1}^\infty (f_{n_{i+1}}(x) - f_{n_i}(x)) & \text{where it converges;} \\ 0 & \text{otherwise.} \end{cases}$$

The partial sum

$$f_{n_1}(x) + \sum_{i=1}^{k-1}(f_{n_{i+1}}(x) - f_{n_i}(x)) = f_{n_k}(x),$$

and so

$$\lim_{k\to\infty} f_{n_k}(x) = f(x) \text{ a.e.}$$

Thus we have shown that every Cauchy sequence has a convergent subsequence, and we NTS that $f_{n_k} \to f$ in L^p.

Given $\epsilon > 0$, there exists N such that $\|f_n - f_m\|_p < \epsilon$ for all $n, m > N$. We have that

$$|f - f_m|^p = \liminf |f_{n_k} - f_m|^p$$

since $f_{n_k} \to f$ a.e. Thus,

$$\begin{aligned}\int_X |f - f_m|^p &= \int_X \liminf |f_{n_k} - f_m|^p \\ &\leq \liminf \int_X |f_{n_k} - f_m|^p \\ &< \epsilon^p.\end{aligned}$$

This implies that $\|f - f_m\|_p < \epsilon$, and thus

$$\|f\|_p = \|f - f_m + f + m\|_p \leq \|f - f_m\|_p + \|f_m\|_p < \infty.$$

We conclude that $f \in L^p$ and $\|f - f_m\|_p \to 0$ as $m \to \infty$.

Now let $p = \infty$ and let $\{f_n\}$ be a Cauchy sequence in $L^\infty(\mu)$. Let

$$A_k = \{x \mid |f_k(x)| > \|f_k\|_\infty\}$$

and

$$B_{m,n} = \{x \mid |f_n(x) - f_m(x)| > \|f_n - f_m\|_\infty\}.$$

These sets all have measure zero. Let

$$N = \left(\bigcup_{k=1}^\infty A_k\right) \cup \left(\bigcup_{n,m=1}^\infty B_{m,n}\right).$$

Then N has measure zero.

For $x \in N^c$, f_n is a Cauchy sequence of complex numbers. Thus, $f_n \to f$ by completeness of $\mathbb{C}$ uniformly. Since $\|f_k\|_\infty$ is bounded, $|f_k(x)| < M$ for all $x \in N^c$. Thus, $f(x) < M$ for all $x \in N^c$. Letting $f = 0$ on N, we have $\|f\|_\infty < \infty$ and $\|f_n - f\|_\infty \to 0$ as $n \to \infty$. □

Theorem 0.6. *Let $1 \leq p \leq \infty$ and $\{f_n\}$ be a Cauchy sequence in $L^p(\mu)$ such that $\|f - f_n\|_p \to 0$. Then f_n has a subsequence which converges pointwise almost everywhere to $f(x)$.*

Proof. Since $\|f - f_n\|_p \to 0$, $f_n \to f$ in measure. By the previous theorem, there exists a subsequence which converges a.e. □

Examples in $\mathbb{R}$.

(1) A sequence in L^p can converge a.e. without converging in L^p. Let $f_k = k^2 \chi_{(0,1/k)}$. Then

$$\|f_k\|_p = \left(\int_{(0,1/k)} k^{2p} \right)^{1/p} = k^2 (1/k)^{1/p} = k^{2-1/p} < \infty.$$

Thus $f_k \in L^p$ and $f_k \to 0$ on $\mathbb{R}$, but $\|f_k\|_p \to \infty$.

(2) A sequence can converge in L^p without converging a.e. (HW problem).

(3) A sequence can belong to $L^{p_1} \cap L^{p_2}$ and converge in L^{p_1} without converging in L^{p_2}. Let $f_k = k^{-1} \chi_{(k,2k)}$. Then $f_k \to 0$ pointwise and $\|f_k\|_p = k^{-1} k^{1/p} = k^{1/p-1}$. If $p > 1$, then $\|f_k\|_p \to 0$ as $k \to \infty$, so $f_k \to 0$ in L^p norm. But $\|f_k\|_1 = 1$ so $f_k \not\to 0$ in L^1.