MEASURE AND INTEGRATION: LECTURE 14

Convex functions. Let $\varphi\colon (a,b) \to \mathbb{R}$, where $-\infty \le a < b \le \infty$. Then φ is *convex* if $\varphi((1-t)x + ty) \le (1-t)\varphi(x) + t\varphi(y)$ for all $x, y \in (a,b)$ and $t \in [0,1]$. Looking at the graph of φ, this means that $(t, \varphi(t))$ lies below the line segment connecting $(x, \varphi(x))$ and $(y, \varphi(y))$ for $x < t < y$.

Convexity is equivalent to the following. For $a < s < t < u < b$,

$$\frac{\varphi(t) - \varphi(s)}{t - s} \le \frac{\varphi(u) - \varphi(t)}{u - t}.$$

If φ is differentiable, then φ is convex on (a,b) if and only if, for $a < s < t < b$, $\varphi'(s) \le \varphi'(t)$. If φ is C^2 (continuously twice differentiable), then φ' increasing $\Rightarrow \varphi'' \ge 0$.

Theorem 0.1. *If φ is convex on (a,b), then φ is continuous on (a,b).*

Jensen's inequality. Let $(\Omega, \mathcal{M}, \mu)$ be a measure space such that $\mu(\Omega) = 1$ (i.e., μ is a probability measure). Let $f\colon \Omega \to \mathbb{R}$ and $f \in L^1(\mu)$. If $a < f(x) < b$ for all $x \in \Omega$ and φ is convex on (a,b), then

$$\varphi\left(\int_\Omega f \, d\mu\right) \le \int_\Omega (\varphi \circ f) d\mu.$$

Proof. Let $t = \int_\Omega f \, d\mu$. Since $a < f < b$,

$$a = a \cdot \mu(\Omega) < \int_\Omega f \, d\mu < b \cdot \mu(\Omega) = b,$$

so $a < t < b$. Conversely,

$$\frac{\varphi(t) - \varphi(s)}{t - s} \le \frac{\varphi(u) - \varphi(t)}{u - t}.$$

Fix t, and let

$$B = \sup_{a<s<t} \frac{\varphi(t) - \varphi(s)}{t - s}.$$

Then $\varphi(t) - \varphi(s) \le B(t - s)$ for $s < t$. We have

$$B \le \frac{\varphi(u) - \varphi(t)}{u - t}$$

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for any $u \in (t, b)$, so $B(u - t) \le \varphi(u) - \varphi(t)$ for $u > t$. Thus $\varphi(s) \ge \varphi(t) + B(s - t)$ for any $a < s < b$. Let $s = f(x)$ for any $x \in \Omega$. Then

$$\varphi(f(x)) - \varphi(t) - B(f(x) - t) \ge 0$$

for all $x \in \Omega$.

Now φ convex $\Rightarrow \varphi$ continuous, so $\varphi \circ f$ is measurable. Thus, integrating with respect to μ,

$$\int_X (\varphi \circ f) d\mu - \int_X \varphi(t)\ d\mu - B \int_X f\ d\mu \ge 0,$$

and the inequality follows. □

Examples.

(1) Let $\varphi(x) = e^x$ be a convex function. Then

$$\exp\left(\int_\Omega f\ d\mu\right) \le \int_\Omega e^f\ d\mu.$$

(2) Let $\Omega = \{p_1, \dots, p_n\}$ be a finite set of points and define $\mu(\{p_i\}) = 1/n$. Then $\mu(\Omega) = 1$. Let $f : \Omega \to \mathbb{R}$ with $f(p_i) = x_i$. Then

$$\int_\Omega f d\mu = \sum_{i=1}^n f(p_i)\mu(\{p_i\})$$

$$= \frac{1}{n}(x_1 + \cdots + x_n).$$

Thus

$$\exp\left(\frac{1}{n}(x_1 + \cdots + x_n)\right) \le \int_\Omega e^f\ d\mu$$

$$\le \frac{1}{n}(e^{x_1} + \cdots + e^{x_n}).$$

Let $y_i = e^{x_i}$. Then

$$(y_1 + \cdots + y_n)^{1/n} \le \frac{1}{n}(y_1 + \cdots + y_n),$$

which is the inequality between arithmetic and geometric means. We also could take $\mu(\{p_i\}) = \alpha_i > 0$ and $\sum_{i=1}^n \alpha_i = 1$. Then

$$y_1^{\alpha_1} y_2^{\alpha_2} \cdots y_n^{\alpha_n} \le \alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_n y_n.$$

Hölder's and Minkowski's inequalities. We define numbers p and q to be *conjugate exponents* if $1/p + 1/q = 1$. The conjugate exponent of 1 is ∞. Conjugate exponents are the same if and only if $p = q = 2$.

Theorem 0.2. *Let p and q be conjugate exponents with $1 < p < \infty$. Let $(X, \mathcal{M}, \mu)$ be a measure space and $f, g \colon X \to [0, \infty]$ measurable functions. Then*

$$\int_X fg \, d\mu \leq \left(\int_X f^p \, d\mu \right)^{1/p} \left(\int_X g^q \, d\mu \right)^{1/q} \qquad \text{(Hölder's)}$$

and

$$\left(\int_X (f+g)^p d\mu \right)^{1/p} \leq \left(\int_X f^p \, d\mu \right)^{1/p} + \left(\int_X g^p \, d\mu \right)^{1/p} \text{(Minkowski's).}$$

Proof. Hölder's. Without loss of generality we may assume that $\int_X f^p = 1$ and $\int_X g^q = 1$. Indeed, if $\int f^p \neq 0$ and $\int g^q \neq 0$, then let

$$\overline{f} = \frac{f}{\left(\int_X f^p\right)^{1/p}}, \quad \overline{g} = \frac{g}{\left(\int_X f^p\right)^{1/p}}.$$

(Otherwise, if $\int f^p = 0$, then $f^p = 0$ a.e., and both sides of the inequality are equal to zero.) We claim that

$$ab \leq \frac{1}{p} a^p + \frac{1}{q} b^q \text{ for all } a, b \in [0, \infty]. \tag{0.1}$$

It is easy to check if a or b equals 0 or ∞. Assume $0 < a < \infty$ and $0 < b < \infty$, and write $a = e^{s/p}$ and $b = e^{t/q}$ for some $s, t \in \mathbb{R}$. Let $\Omega = \{x_1, x_2\}$, $\mu(x_1) = 1/p$, and $\mu(x_2) = 1/q$. We have

$$\exp \int_\Omega f \, d\mu \leq \int_\Omega e^f \, d\mu,$$

where $f(x_1) = s$ and $f(x_2) = t$. Thus,

$$\exp\left(\frac{s}{p} + \frac{t}{q} \right) \leq \frac{1}{p} e^s + \frac{1}{q} e^t,$$

so (0.1) follows. Thus,

$$\int_X (fg) d\mu \leq \frac{1}{p} \int_X a^p \, d\mu + \frac{1}{q} \int_X b^q \, d\mu = \frac{1}{p} + \frac{1}{q} = 1.$$

Minkowski's. Observe that

$$(f+g)^p = f(f+g)^{p-1} + g(f+g)^{p-1}.$$

Since p and q are conjugate exponents, $q = p/(p-1)$. Thus,

$$\begin{aligned}\int f(f+g)^{p-1} &\leq \left(\int f^p\right)^{1/p} \left(\int (f+g)^{(p-1)p/(p-1)}\right)^{(p-1)/p} \\ &= \left(\int f^p\right)^{1/p} \left(\int (f+g)^p\right)^{(p-1)/p}.\end{aligned}$$

Similarly,

$$\int f(f+g)^{p-1} \leq \left(\int f^p\right)^{1/p} \left(\int (f+g)^p\right)^{(p-1)/p}.$$

Let $\Omega = \{x_1, x_2\}$, $\mu(x_1) = 1/2 = \mu(x_2)$, and $\varphi = t^p$. Then

$$\left(\int_\Omega f \; d\mu\right)^p \leq \int_\Omega f^p \; d\mu,$$

so

$$\left(\frac{a+b}{2}\right)^p \leq \frac{a^p}{2} + \frac{b^p}{2}.$$

Thus,

$$\frac{1}{2^p}\int (f+g)^p \leq \frac{1}{2}\int f^p + \frac{1}{2}\int g^p < \infty.$$

Since $1 - (p-1)/p = 1/p$,

$$\left(\int_X (f+g)^p d\mu\right)^{1/p} \leq \left(\int_X f^p \; d\mu\right)^{1/p} + \left(\int_X g^p \; d\mu\right)^{1/p}.$$

□