MEASURE AND INTEGRATION: LECTURE 12

Appoximation of measurable functions by continuous functions. Recall Lusin's theorem.Let $f\colon X \to \mathbb{C}$ be measurable, $A \subset X$, $\mu(A) < \infty$, and $f(x) = 0$ if $x \notin A$. Given $\epsilon > 0$, there exists $g \in C_c(X)$ such that $\mu(\{x \mid f(x) \neq g(x)\}) < \epsilon$ and

$$\sup_{x \in X} |g(x)| \leq \sup_{x \in X} |f(x)| .$$

A corollary with the same assumptions and f bounded (i.e., $|f(x)| < M$) is that there exists sequence $g_n \in C_c(X)$, $|g_n| < M$ such that $\lim g_n(x) = f(x)$ almost everywhere.

Convergence almost everywhere. Lebesgue's dominated convergence theorem (LDCT) in the case of almost everywhere.

Theorem 0.1. *Let $f_1, f_2, \ldots : X \to \mathbb{C}$ be a sequence of measurable functions defined a.e. Let $g\colon X \to \mathbb{C}$ be defined almost everywhere and $g \in L^1(\mu)$. Assume $\lim_{k\to\infty} f_k(x)$ exists for a.e. $x \in X$ and $|f_k(x)| \leq |g(x)|$ for a.e. $x \in X$. Then*

$$\int_X \left(\lim_{k\to\infty} f_k\right) d\mu = \lim_{k\to\infty} \int_X f_k \; d\mu.$$

Proof. Let $E_k = \{x \mid |f_k(x)| \geq |g(x)|\}$. Then $\mu(E_k) = 0$. Let $E = \cup_{k=1}^{\infty} E_k$. Then $\mu(E) = 0$. Redefine $f_k = 0$ on E; this does not change the integrals. Now $|f_k| \leq |g|$ a.e., and we can apply the regular LDCT. □

Theorem 0.2. *Let $f_1, f_2, \ldots : X \to \mathbb{C}$ with each $f_k \in L^1(\mu)$ and assume that $\sum_{k=1}^{\infty} \int_X |f_k| \, d\mu < \infty$. Then $\sum_{k=1}^{\infty} f_k$ exists a.e. and*

$$\int_X \left(\sum_{k=1}^{\infty} f_k\right) d\mu = \sum_{k=1}^{\infty} \int_X f_k \; d\mu.$$

Proof. Let $g = \sum_{k=1}^{\infty} |f_k|$. Monotone convergence implies that $\int g = \int \sum_{k=1}^{\infty} |f_k| = \sum_{k=1}^{\infty} \int |f_k| < \infty$. Thus $g \in L^1(\mu)$ and so $g < \infty$ a.e. Thus, $\sum_{k=1}^{\infty} |f_k(x)| < \infty$ a.e. This implies that the series $\sum_{k=1}^{\infty} f_k(x)$

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converges absolutely a.e. Let $F_j = \sum_{k=1}^{j} f_k$. Then F_j is dominated by g for all j, and we can apply LDCT. We have

$$\begin{aligned}\int_X \left(\sum_{k=1}^{\infty} f_k\right) d\mu &= \int_X \lim_{j\to\infty} F_j \; d\mu = \lim_{j\to\infty} \int_X F_j \; d\mu \\ &= \lim_{j\to\infty} \int_X \sum_{k=1}^{j} f_k \; d\mu = \lim_{j\to\infty} \sum_{k=1}^{j} \int_X f_k \; d\mu \\ &= \sum_{k=1}^{\infty} \int_X f_k \; d\mu.\end{aligned}$$

□

Countable additivity of the integral. Let $E_1, E_2, \ldots$ be a countable sequence of measurable sets. Let $E = \cup_{k=1}^{\infty} E_k$ and $f\colon X \to \mathbb{C}$ be measurable. Assume either $f \geq 0$ or $f \in L^1(E)$ (i.e., $\int_E f \; d\mu = \int_X f\chi_E \; d\mu < \infty$). Then

$$\int_E f \; d\mu = \sum_{k=1}^{\infty} \int_{E_k} f \; d\mu.$$

Proof. First let $f \geq 0$. Then

$$\begin{aligned}\int_E f \; d\mu &= \int_X f \; \chi_E \; d\mu \\ &= \int_E \sum_{k=1}^{\infty} f \; \chi_{E_k} \; d\mu\end{aligned}$$

$$\sum_{k=1}^{\infty} \int_X f \; \chi_{E_k} \; d\mu$$

$$= \sum_{k=1}^{\infty} \int_{Ek} f \; d\mu.$$

Now let $f \in L^1(E)$ and $f_k = \chi_{E_k}$. By the previous theorem, we need only check the convergence of the series of integrals of $|f\chi_{E_k}|$.

We have

$$\begin{aligned}\sum_{k=1}^{\infty} \int_X |f_k| \; d\mu &= \sum_{k=1}^{\infty} \int_X |f| \, \chi_{E_k} \; d\mu \\ &= \sum_{k=1}^{\infty} \int_{E_k} |f| \; d\mu \\ &= \int_E |f| \; d\mu < \infty,\end{aligned}$$

because of the case when $f \geq 0$.

□