MEASURE AND INTEGRATION: LECTURE 11

Principles of measure theory.

(1) Every measurable set is nearly a Borel set: $A = F_\sigma \cup N = G_\delta \cup N$ (N is a null set: a set of measure zero).
(2) Every measurable set is nearly an open set: $\lambda(U) < \lambda(E) + \epsilon$.
(3) Every measurable function is nearly continuous (Lusin's theorem).
(4) Every convergent sequence of measurable functions is nearly uniformly convergent (Egoroff's theorem).

Lusin's theorem.

Theorem 0.1. *Let $f\colon X \to \mathbb{R}$ (or $\mathbb{C}$) be a measurable function on a locally compact Hausdorff space X. Let $A \subset X$, $\mu(A) < \infty$, and $f(x) = 0$ if $x \notin A$. Given $\epsilon > 0$, there exists $g \in C_c(X)$ such that $\mu(\{x \mid f(x) \neq g(x)\}) < \epsilon$.*

Proof. Assume that $0 \leq f \leq 1$ and A compact. Recall that if $f\colon X \to [0,\infty]$ is measurable, then there exist simple measurable functions s_i such that (a) $0 \leq s_1 \leq \cdots \leq f$ and (b) $s_i \to f$ as $i \to \infty$. (Proof: Let $\delta_= 2^{-n}$ and for $t \geq 0$ define $k_n(t)$ such that $k\delta_n \leq t < (k+1)\delta_n$, and define

$$\varphi_n(t) = \begin{cases} k_n(t)\delta_n & 0 \leq t < n; \\ n & n \leq t \leq \infty. \end{cases}$$

Then $\varphi_n(t) \leq t$ and $\varphi_n(t) \to t$ as $n \to \infty$. The function $\varphi \circ f$ is simple and $\varphi_n \circ f \to f$ as $n \to \infty$.)

Next, let $t_1 = s_1, \ldots, t_n = s_n - s_{n-1}$. Claim: $s_n - s_{n-1}$ takes only values 0 and 2^{-n}. Let $T_n \subset A$ where $T_n = \{x \mid t_n = 2^{-n}\}$. Then

$$f(x) = s_1 + (s_2 - s_1) + \cdots = \sum_{n=1}^{\infty} t_n.$$

Since X is locally compact, we may choose $A \subset V$, V open, and $\overline{V}$ compact. There exists $K_n \subset T_n \subset V_n \subset V$, K_n compact, V_n open, such that $\mu(V_n \setminus K_n) < 2^{-n}\epsilon$. By Urysohn's lemma, there exist function h_n such that $K_n \prec h_n \prec V_n$. Define $g(x) = \sum_{n=1}^{\infty} 2^{-n} h_n(x)$. Since

Date: October 9, 2003.

this series converges uniformly on X, g is continuous and $\operatorname{supp} f \subset \overline{V}$. But $2^{-n}h_n(X) = t_n$ except on $V_n \setminus K_n$. Thus, $g(x) = f(x)$ except on $\cup_{n=1}^{\infty} V_n \setminus K_n$, and μ of this set is less than or equal to $\sum_{n=1}^{\infty} \epsilon/2^n = \epsilon$. Thus, we have proved the case where $0 \leq f \leq 1$ and A is compact. Thus, it is true when f is a bounded measurable function and A is compact.

Now look at

$$\overline{f} = \frac{f}{(\sup f) + 1}.$$

If A is not compact and $\mu(A) < \infty$, then there exists $K \subset A$ such that $\mu(A \setminus K) < \epsilon$ for any ϵ. Let $g = 0$ on $A \setminus K$. For f not bounded, let $B_n = \{x \mid |f(x)| > n\}$. Then $\cap_n B_n = \emptyset$, so $\mu(B_n) \to 0$. Then f agrees with $(1 - \chi_{B_n})f$ except on B_n, and we can let $g = 0$ on B_n. □

Corollary 0.2. *Let $f\colon X \to \mathbb{R}$, $A \subset X$, $\mu(A) < \infty$, $f(x) = 0$ if $x \notin A$, and $|f(x)| < M$ for some $M < \infty$. Then there exists a sequence $g_n \in C_c(X)$ such that $|g_n(x)| < M$ and $f(x) = \lim_{n\to\infty} g_n(x)$ almost everywhere.*

Proof. By the theorem, for $n > 0$, there exists $g_n \in C_c(X)$ such that $\sup |g_n| \leq \sup |f| < M$ and $\mu(E_n) < 2^{-n}$. Let $E_n = \{x \mid f \neq g_n\}$. Then E_n is measurable and $\sum_{n=1}^{\infty} \mu(E_n) = \sum_{n=1}^{\infty} 2^{-n} < \infty$. Claim: almost all $x \in X$ lie in at most finitely many E_k. Proof: Let $g(x) = \sum_{k=1}^{\infty} \chi_{E_k}(x)$. Then x is in infinitely many $E_k \iff g(x) = \infty$. We have

$$\int_X g(x)\ d\mu = \sum_{k=1}^{\infty} \int_X \chi_{E_k}(x)\ d\mu < \infty$$

by monotone convergence. Thus, $g(x) < \infty$ almost everywhere, which implies that $\lim g_n = f$ a.e. □

Vitali-Caratheodory theorem.

Theorem 0.3. *Let $f\colon X \to \mathbb{R}$ and $f \in L^1(\mu)$. Given $\epsilon > 0$, there exists functions $u, v\colon X \to \mathbb{R}$ such that $u \leq f \leq v$, u is upper semicontinuous, v is lower semicontinuous, and $\int (v - u) d\mu < \epsilon$.*

Proof. Assume $f \geq 0$. Choose $0 \leq s_1 \leq \cdots \leq f$ simple and measurable such that $\lim s_n = f$. Let $t_n = s_n - s_{n-1}$. Then t_n has only finitely many values and $f = \sum_{k=1}^{\infty} t_n = \sum_{k=1}^{\infty} c_k \chi_{E_k}$. We have

$$\int_X f\ d\mu = \sum_{i=1}^{\infty} c_i \mu(E_i) < \infty.$$

Choose $K_i \subset E_i \subset V_i$, K_i compact, V_i open, such that $c_i\mu(V_i \setminus K_i) < 2^{-i-1}\epsilon$. Let

$$v = \sum_{i=1}^{\infty} c_i \chi_{V_i}, \quad u = \sum_{i=1}^{N} c_i \chi_{K_i},$$

where N is chosen so that $\sum_{N+1}^{\infty} c_i\mu(E_i) < \epsilon/2$. Then v is lower semicontinuous, u is upper semicontinuous, and $u \le f \le v$. Also,

$$\begin{aligned} v - u &= \sum_{i=1}^{N} c_i(\chi_{V_i} - \chi_{K_i}) + \sum_{N+1}^{\infty} c_i \chi_{V_i} \\ &\le \sum_{i=1}^{\infty} c_i(\chi_{V_i} - \chi_{K_i}) + \sum_{i=N+1}^{\infty} c_i(\chi_{V_i} - \chi_{V_i} + \chi_{K_i}) \\ &\le \sum_{i=N+1}^{\infty} c_i \chi_{K_i}, \end{aligned}$$

and so

$$\begin{aligned} \int_X (v-u)d\mu &\le \sum_{i=1}^{\infty} c_i \mu(V_i \setminus K_i) + \sum_{N+1}^{\infty} c_i \chi_{E_i} \\ &< \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

□