MEASURE AND INTEGRATION: LECTURE 10

Integration as a linear functional. A complex vector space is a set V with two operations: addition $(+)$ and scalar multiplication $(\cdot)$.

Addition: For all $x, y, z \in V$,

- $x + y = y + x$.
- $x + (y + z) = (x + y) + z$.
- $\exists$ unique vector 0 such that $x + 0 = x$ for all x.
- $\exists$ $(-x)$ such that $x + (-x) = 0$.

Multiplication: For all $\alpha, \beta \in \mathbb{C}$, $x \in V$,

- $1 \cdot x = x$
- $\alpha \cdot (\beta \cdot x) = (\alpha\beta) \cdot x$
- $\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y$
- $(\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot x$.

A linear transformation is a map $\Lambda\colon V_1 \to V_2$ from a vector space V_1 to a vector space V_2 such that $\Lambda(\alpha x_\beta y) = \alpha\Lambda x + \beta\Lambda y$. If $V_2 = \mathbb{C}$ (or $\mathbb{R}$), then Λ is a *linear functional.*

Let $(X, \mathcal{M}, \mu)$ be a measure space. Then

$$L^1(\mu) = \left\{ f\colon X \to \mathbb{C} \mid \int_X |f|\, d\mu < \infty, f \text{ measurable} \right\}.$$

Note that $\int_X\colon f \mapsto \int_X f\ d\mu$ is a linear functional. Let $g\colon X \to \mathbb{C}$ be a bounded measurable function. Then $f \mapsto \int_X fg\ d\mu$ is also a linear functional.

Special case: $X = \mathbb{R}^n$. Let

$$C(\mathbb{R}^n, \mathbb{R}) = \{f\colon \mathbb{R}^n \to \mathbb{R} \mid f \text{ continuous}\}.$$

The Riemann integral is a *positive* linear functional since $f \geq 0 \Rightarrow \Lambda f \geq 0$, where Λ is the Riemann integral.

Riesz theorem. Let X be a topological space and $C(X)$ be the set of functions from X to $\mathbb{R}$. If $\Lambda\colon C \to \mathbb{R}$ is a positive linear functional, then there exists a σ-algebra $\mathcal{M}$ and unique measure μ on X such that $\Lambda f = \int_X f\ d\mu$. Conversely, given a measure, then Λ is a positive linear functional.

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Topology. Let X be a topological space. The space X is *Hausdorff* if for all $p, q \in X$ such that $p \neq q$ there exist neighborhoods U and V such that $p \in U$, $q \in V$, and $U \cap V = \emptyset$. The space X is *locally compact* if for all $p \in X$ there exists a neighborhood U of p such that $\overline{U}$ (the closure of U) is compact. (Infinite dimensional spaces are not locally compact.)

Let $f\colon X \to \mathbb{R}$. If $\{x \mid f(x) > \alpha\}$ is open for all α, then f is *lower semicontinuous*. If $\{x \mid f(x) < \alpha\}$ is open for all α, then f is *upper semicontinuous*. Examples: χ_U for U open is lower semicontinuous and χ_F for F closed is upper semicontinuous.

The *support* of a function f is defined as the set $\operatorname{supp} f = \{x \mid f(x) \neq 0\}$. An important set is the set of all functions with compact support:

$$C_c(X) = \{f\colon X \to \mathbb{C} \mid \operatorname{supp} f \text{ is compact}\}.$$

Since $\operatorname{supp} f_g \subset (\operatorname{supp} f) \cup (\operatorname{supp} g)$, $C_c(X)$ is a vector space.

Notation: (1) $K \prec f$ means that K is compact, $f \in \mathbb{C}_c(X)$, $0 \leq f(x) \leq 1$ for all $x \in X$, and $f(x) = 1$ for all $x \in K$. (2) $f \prec V$ means that V is open, $f \in C_c(X)$, $0 \leq f(x) \leq 1$ for all $x \in X$, and $\operatorname{supp} f \subset V$.

Urysohn's lemma. Let X be a locally compact Hausdorff space, $K \subset V$, K compact, U open. Then there exists $f \in C_c(X)$ such that $K \prec f \prec V$.

A corollary to Urysohn's lemma is the existence of partitions of unity. Let $V_1, \ldots, V_n$ be open subsets of X (a locally compact Hausdorff space) and K compact such that $K \subset V_1 \cup \cdots \cup V_n$. Then there exists functions $h_i \prec V_i$ such that $h_1(x) + \cdots + h_n(x) = 1$.

Riesz representation theorem (for positive linear functionals).

Theorem 0.1. *Let X be a locally compact Hausdorff space. Let*

$$\Lambda\colon C_c(X) \to \mathbb{C}$$

be a positive linear functional (positive when restricted to $f\colon X \to \mathbb{R}_{\geq 0}$). Then there exists a σ-algebra $\mathcal{M}$ in X which contains all the Borel sets and a unique positive measure μ on $\mathcal{M}$ such that

(a) *$\Lambda f = \int_X f\, d\mu$ for all $f \in C_c(X)$.*
(b) *$\mu(K) < \infty$ for all compact sets $K \subset X$.*
(c) *If $E \in \mathcal{M}$, then*

$$\mu(E) = \inf\{\mu(V) \mid E \subset V, V \text{ open}\}.$$

(d) *If E is open or $E \in \mathcal{M}$ with $\mu(E) < \infty$, then*

$$\mu(E) = \sup\{\mu(K) \mid K \subset E, K \text{ compact}\}.$$

(e) *If $E \in \mathcal{M}$, $A \subset E$, and $\mu(E) = 0$, then $A \in \mathcal{M}$.*

Proof. (Outline) We must show uniqueness.

By (d), the measure of open sets determined by measure of compact sets, and so by (c) the measure of any set in $\mathcal{M}$ is determined by the measure of compact sets. Assume we have μ_1 and μ_2 which satisfy the conditions of the theorem, and let K be compact. For any $\epsilon > 0$, choose U open such that $K \subset U$ and $\mu_2(U) < \mu_2(K)+\epsilon$. By Urysohn's lemma, there exists $f \in C_c(X)$ such that $K \prec f \prec V$. Then

$$\mu_1(K) = \int_X \chi_K \, d\mu_1 \leq \int_X f \, d\mu_1 = \Lambda f$$

and

$$\Lambda f = \int_X f \, d\mu_2 \leq \int_X \chi_V \, d\mu_2 = \mu_2(V) < \mu_2(K) + \epsilon.$$

Since this holds for any $\epsilon > 0$, $\mu_1(K) \leq \mu_2(K)$, and by reversing the roles of μ_1 and μ_2, we have $\mu_1(K) = \mu_2(K)$.

Now let $V \subset X$ be open and define $\mu(V) = \sup\{\Lambda f \mid f \prec V\}$. For $E \subset X$, define $\mu(E) = \inf\{\mu(V) \mid E \subset V, V \text{ open}\} = \lambda^*(E)$. ($\lambda^*$ will not be countably additive on all sets, only on the σ-algebra.) Let $\mathcal{M}_F$ be the set of $E \subset X$ such that

$$\mu(E) = \sup\{\mu(K) \mid K \subset E, K \text{ compact}\} \text{ and } \mu^*(E) < \infty.$$

Finally, $\mathcal{M}$ is simply $E \subset X$ such that $E \cap K \in \mathcal{M}_F$ for all $K \in \mathcal{M}_F$. □

Properties.

1. μ^* is countably subadditive: $\mu(\cup E_i) \leq \sum \mu(E_i)$.
2. If $E_i \in \mathcal{M}_F$ are disjoint, then $\mu(\cup E_i) = \sum \mu(E_i)$.
3. M_F contains all open sets.
4. (Approximation) If $E \in \mathcal{M}_F$ and $\epsilon > 0$, then there exist $K \subset E \subset V$, K compact, V open, such that $\mu(V \setminus K) < \epsilon$.
5. $\mathcal{M}$ is a σ-algebra that contains the Borel σ-algebra $\mathcal{B}$, and μ is countably additive on $\mathcal{M}$.
6. If $f \in C_c(X)$, then $\Lambda f = \int_X f \, d\mu$.

   Proof. Just NTS that $\Lambda f \leq \int_X f \, d\mu$ for f real in $C_c(X)$. Then

   $$-\Lambda f = \Lambda(-f) \leq \int_X (-f) d\mu = -\int_X f \, d\mu$$

   $$\Rightarrow \Lambda f \geq \int_X f \, d\mu.$$

   The complex case follows from the real case by complex linearity. Let $f \in C_c(X)$ and $\operatorname{supp} f = K$ compact. The continuous image of compact sets is compact $\Rightarrow f(K) \subset [a,b]$. Choose $\epsilon > 0$ and choose y_i $(i = 0, 1, \ldots, n)$ such that $y_i - y_{i=1} < \epsilon$ and $y_0 < a < y_1 \cdots < y_n = b$ (i.e., partition the range by ϵ). Let

$$E_i = \{x \mid y_{i=1} < f(x) \leq y_i\} \cap K.$$

Since f is continuous, f is Borel measurable and $\cup_{i=1}^n E_i = K$ is a disjoint union. choose open sets $V_i \supset E_i$ such that $\mu(V_i) < \mu(E_i) + \epsilon/n$ for each $i = 1, \ldots, n$ and $f(x) < y_i + \epsilon$ for all $x \in V_i$. (The latter can be done by continuity of f.)

By partition of unity, there exists $h_i \prec V_i$ such that $\sum_i h_i = 1$ on K. Write $f = \sum_i h_i f$. Then

$$\mu(K) \leq \Lambda(\sum_i h_i) = \sum_i \Lambda h_i,$$

$$h_i f \leq (y_i + \epsilon) h_i, \text{ and } y_i - \epsilon < f(x) \ \forall x \in E_i.$$

Thus,

$$\begin{aligned}
\Lambda f &= \sum_{i=1}^n \Lambda(h_i f) \leq \sum_{i=1}^n (y_i + \epsilon)\Lambda h_i \\
&= \sum_{i=1}^n (|a| + y_i + \epsilon)\Lambda h_i - |a| \sum_{i=1}^n \Lambda h_i \\
&\leq \sum_{i=1}^n (|a| + y_i + \epsilon)(\mu(E_i) + \epsilon/n) - |a|\, \mu(K) \\
&= \sum_{i=1}^n (|a| + \epsilon)(\mu(E_i)) \\
&= |a|\, \mu(K) + \sum_{i=1}^n (|a| + y_i + \epsilon)\epsilon/n + \sum_{i=1}^n y_i \mu(E_i) \\
&= \sum_{i=1}^n (y_i - \epsilon)\mu(E_i) + 2\epsilon\mu(K) + \epsilon/n \sum_{i=1}^n (|a| + y_i + \epsilon) \\
&\leq \int_X f \, d\mu + \epsilon(\text{constant}).
\end{aligned}$$

□

Definitions. A measure space $(X, \mathcal{M}, \mu)$ is called a *Borel measure* if $\mathcal{B} \subset \mathcal{M}$. If $\mu(E) = \inf\{\mu(V) \mid E \subset V, V \text{ open}\}$ for all $E \in \mathcal{M}$, then μ is called *outer regular*. Similarly, if $\mu(E) = \sup\{\mu(K) \mid K \subset E, K \text{ compact}\}$ for all $E \in \mathcal{M}$, then μ is called *inner regular*. If μ is both inner and outer regular, it is said to be *regular*.

A space X is σ-compact if $X = \cup_{i=1}^{\infty} K_i$ where each K_i is compact. It is σ-finite if $X = \cup_{i=1}^{\infty} E_i$ where $\mu(E_i) < \infty$ for each i.

Addition to Riesz. If X is locally compact, σ-compact, Hausdorff space then we also have:

(1) If $E \in \mathcal{M}$ and $\epsilon > 0$, then there exists $F \subset E \subset V$, F closed, V open, such that $\mu(V \setminus F) < \epsilon$.
(2) For all $E \in \mathcal{M}$ there exists $A \subset E \subset B$ such that A is F_σ, B is G_δ, and $\mu(B \setminus A) = 0$.

Application. Let $X = \mathbb{R}^k$, $\Lambda\colon C_c(X) \to \mathbb{R}$ given by $\Lambda f = \int_X f$, the Riemann integral. Then Lebesgue measure is what you get from the Riesz theorem.