MEASURE AND INTEGRATION: LECTURE 1

Preliminaries. We need to know how to measure the "size" or "volume" of subsets of a space X before we can integrate functions $f\colon X \to \mathbb{R}$ or $f\colon X \to \mathbb{C}$.

We're familiar with volume in $\mathbb{R}^n$. What about more general spaces X? We need a measure function $\mu\colon \{\text{subsets of } X\} \to [0, \infty]$.

For technical reasons, a measure will not be defined on *all* subsets of X, but instead a certain collection of subsets of X called a σ-algebra, a collection of subsets of X (i.e., a collection $\mathcal{M} \subset \mathcal{P}(X)$ that is a subset of the power set of X) satisfying the following:

(σ1) $X \in \mathcal{M}$.
(σ2) If $A \in \mathcal{M}$, then $A^c \equiv X \setminus A \in \mathcal{M}$.
(σ3) If $A_i \in \mathcal{M}$ $(i = 1, 2, \ldots)$, then $\cup_{i=1}^{\infty} \in \mathcal{M}$.

Constrast with a topology $\tau \subset \mathcal{P}(X)$, which satisfies

(τ1) $\emptyset \in \tau$ and $X \in \tau$.
(τ2) If $U_i \in \tau$ $(i = 1, \ldots, n)$, then $\cap_{i=1}^{n} U_i \in \tau$.
(τ3) If U_α $(\alpha \in \mathcal{I})$ is an arbitrary collection in τ, then $\cup_{\alpha \in \mathcal{I}} U_\alpha \in \tau$.

Remarks on σ-algebras:

(a) By (σ1), $X \in \mathcal{M}$, so by (σ2), $\emptyset \in \mathcal{M}$.
(b) $\cap_{i=1}^{\infty} A_i = (\cup_{i=1}^{\infty} A_i^c)^c \Rightarrow$ countable intersections are in $\mathcal{M}$.
(c) $A, B \in \mathcal{M} \Rightarrow A \setminus B \in \mathcal{M}$ (since $A \setminus B = A \cap B^c$).

Let (X, τ_X) and (Y, τ_Y) be a topological spaces. Then $f\colon X \to Y$ is *continuous* if $f^{-1}(U) \in \tau_X$ for all $U \in \tau_Y$. "Inverse images of open sets are open."

Let $(X, \mathcal{M})$ be a measure space (i.e., $\mathcal{M}$ is a σ-algebra for the space X). Then $f\colon X \to Y$ is *measurable* if $f^{-1}(U) \in \mathcal{M}$ for all $U \in \tau_Y$. "Inverse images of open sets are measurable."

Basic properties of measurable functions.

Proposition 0.1. *Let X, Y, Z be topological spaces such that*

$$X \xrightarrow{f} Y \xrightarrow{g} Z.$$

(1) *If f and g are continuous, then $g \circ f$ is continuous.*

Proof. $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U)) = f^{-1}(\text{open}) = \text{open}.$ □

Date: September 4, 2003.

(2) *If f is measurable and g is continuous, then $g \circ f$ is measurable.*

Proof. $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U)) = f^{-1}(\text{open}) = \text{open}$. □

Theorem 0.2. *Let $u\colon X \to \mathbb{R}$, $v\colon X \to \mathbb{R}$, and $\Phi\colon \mathbb{R} \times \mathbb{R} \to Y$. Set $h(x) = \Phi(u(x), v(x))\colon X \to Y$. If u and v are measurable and Φ is continuous, then $h\colon X \to Y$ is measurable.*

Proof. Define $f\colon X \to \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ by $f(x) = u(x) \times v(x)$. Then $h = \Phi \dot{f}$. We just need to show (NTS) that f is measurable. Let $R \subset \mathbb{R}^2$ be a rectangle of the form $I_1 \times I_2$ where each $I_i \subset \mathbb{R}(i = 1, 2)$ is an open interval. Then $f^{-1}(R) = u^{-1}(I_1) \cap v^{-1}(I_2)$. Let $x \in f^{-1}(R)$ so that $f(x) \in R$. Then $u(x) \in I_1$ and $v(x) \in I_2$. Since u is measurable, $u^{-1}(I_1) \in \mathcal{M}$, and since v is measurable, $v^{-1}(I_2) \in \mathcal{M}$. Since $\mathcal{M}$ is a σ-algebra, $u^{-1}(I_1) \cap v^{-1}(I_2) \in \mathcal{M}$. Thus $f^{-1}(R) \in \mathcal{M}$ for any rectangle R. Finally, any open set $U = \cup_{i=1}^{\infty} R_i$ (rectangle around points with rational coordinates). So $f^{-1}(U) = f^{-1}\left(\cup_{i=1}^{\infty} R_i\right) = \cup_{i=1}^{\infty} f^{-1}(R_i)$. Each term in the union is in $\mathcal{M}$, so since countable unions of elements in $\mathcal{M}$ are in $\mathcal{M}$, $f^{-1}(U) \in \mathcal{M}$. □

Examples.

(a) Let $f\colon X \to \mathbb{C}$ with $f = u + iv$ and u, v real measurable functions. Then f is complex measurable.

(b) If $f = u + iv$ is complex measurable on X, then u, v, and $|f|$ are real measurable. Take Φ to be $z \mapsto \operatorname{Re} z$, $z \mapsto \operatorname{Im} z$, and $z \mapsto |z|$, respectively.

(c) If f, g are real measurable, then so are $f + g$ and fg. (Also holds for complex measurable functions.)

(d) If $E \subset X$ is measurable (i.e., $E \in \mathcal{M}$), then the characteristic function of E,

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E; \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 0.3. *Let $\mathcal{F}$ be any collection of subsets of X. Then there exists a smallest σ-algebra $\mathcal{M}^*$ such that $\mathcal{F} \subset \mathcal{M}^*$. We call M^* the σ-algebra generated by $\mathcal{F}$.*

Proof. Let $\Omega =$ the set of all σ-algebras containing $\mathcal{F}$. The power set $\mathcal{P}(X) =$ the set of all subsets of X is a σ-algebra, so Ω is not empty. Define $\mathcal{M}^* = \cap_{\mathcal{M} \in \Omega} \mathcal{M}$. Since $\mathcal{F} \in \mathcal{M}$ for all $\mathcal{M} \in \Omega$, we have $\mathcal{F} \subset \mathcal{M}^*$. If $\mathcal{M}$ is a σ-algebra containing $\mathcal{F}$, then $\mathcal{M}^* \subset \mathcal{M}$ by definition. Claim: $\mathcal{M}^*$ is a σ-algebra. If $A \in \mathcal{M}^*$, take $\mathcal{M} \in \Omega$. $\mathcal{M}$ is a σ-algebra and $A \in \mathcal{M}$. Thus, $A^c \in \mathcal{M}$, and so $A^c \in \mathcal{M}^*$ since $\mathcal{M}^* \subset \mathcal{M}$. If $A_i \in \mathcal{M}^*$

for each $i = 1, 2, \ldots$, then $A \in \mathcal{M}$, and so $\cup_i A_i \in \mathcal{M}$. It follows that $\cup_i A_i \in \mathcal{M}^*$. □

Borel Sets. By the previous proposition, if X is a topological space, then there exists a smallest σ-algebra $\mathcal{B}$ containing the open sets. Elements of $\mathcal{B}$ are called *Borel sets*.

If $f\colon (X, \mathcal{B}) \to (Y, \tau)$ and $f^{-1}(U) \in \mathcal{B}$ for all $U \in \tau$, then f is called *Borel measurable*. In particular, continuous functions are Borel measurable.

Terminology:

- F_σ ("F-sigma") = countable union of closed sets.
- G_δ ("G-delta") = countable intersection of open sets.