18.112 Functions of a Complex Variable
Fall 2008

Lecture 7: Linear Transformations

(Text 80-89)

Remarks on Lecture 6

Concerning Definition 13 p. 81, formula (11) shows that the definition does not depend on the choice of z_1, z_2, z_3.

Exercise 2 on page 88 requires a minor correction. For example $w = -z$ is hyperbolic according to definition on page 86, yet when written in the form

$$\frac{az+b}{cz+d}$$

with

$$ad - bc = 1,$$

we must take

$$a = -d = i,$$

so

$$a + d = 0.$$

The transformation $w = z$ causes other ambiguities.

Thus we modify the definition a bit.

Definition 1

- *S is parabolic if either it is the identity or has exactly one fixed point.*
- *S is strictly hyperbolic if $k > 0$ in (12) but $k \neq 1$.*
- *S is elliptic if $|k| = 1$ in (12) p. 86 but S is not identity.*

Then the statement of Exercise 2 holds with hyperbolic replaced by strictly hyperbolic.

(i) the condition for exactly one fixed point for

$$Sz = \frac{\alpha z + \beta}{\gamma z + \delta}$$

is

$$(\alpha - \delta)^2 = -4\beta\gamma \qquad \text{(wrong sign in text).}$$

With the normalization

$$\alpha\delta - \beta\gamma = 1$$

this amounts to

$$(\alpha + \delta)^2 = 4$$

as desired.

(ii) Assume two fixed points are a and b, so

$$\frac{w-a}{w-b} = k\frac{z-a}{z-b},$$

which we write as

$$w = Tz = \frac{\alpha z + \beta}{\gamma z + \delta}.$$

Put

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

and define

$$\mathrm{Tr}^2(T) = \frac{(\mathrm{Trace}A)^2}{\det A}.$$

By linear algebra,

$$\mathrm{Trace}(BAB^{-1}) = \mathrm{Trace}(A)$$

and

$$\det(BAB^{-1}) = \det A.$$

Define

$$z_1 = Sz = \frac{z-a}{z-b},$$

$$w_1 = Sw = \frac{w-a}{w-b}.$$

Then

$$\mathrm{Tr}^2(T) = \mathrm{Tr}^2(STS^{-1}).$$

Now

$$w_1 = STz = STS^{-1}z_1,$$

so

$$w_1 = kz_1.$$

Then

$$\mathrm{Tr}^2(T) = \mathrm{Tr}^2(STS^{-1}) = k + \frac{1}{k} + 2.$$

If T is strictly hyperbolic, we have

$$k > 0, \quad k \neq 1,$$

so

$$\mathrm{Tr}^2(T) > 4,$$

which under the assumption

$$\alpha\delta - \beta\gamma = 1$$

amounts to

$$(\alpha + \delta)^2 > 4$$

as stated.

Conversely, if

$$(\alpha + \delta)^2 > 4,$$

then $k > 0$. So the transformation

$$w_1 = kz_1$$

maps each line through 0 and ∞ into itself. So T maps each circle C_1 into itself with $k > 0$. Thus T is strictly hyperbolic.

(iii) If $|k| = 1$, then

$$w_1 = e^{i\theta} z_1$$

and we find

$$\mathrm{Tr}^2(T) = \left(2\cos\frac{\theta}{2}\right)^2 < 4$$

since the possibility $\theta = 0$ is excluded.

Conversely, if

$$-2 < \alpha + \delta < 2, \quad \alpha\delta - \beta\gamma = 1$$

we have

$$\mathrm{Tr}^2(T) = (\alpha + \delta)^2 = k + \frac{1}{k} + 2 < 4.$$

Writing

$$k = re^{i\theta} \quad (r > 0)$$

this implies

$$\left(r + \frac{1}{r}\right) \cos\theta + i \left(r - \frac{1}{r}\right) \sin\theta < 2,$$

which implies

$$r = 1 \qquad \text{or} \qquad \theta = 0 \qquad \text{or} \qquad \theta = \pi.$$

If $r = 1$, then $|k| = 1$, so T is elliptic.
Since $r + \frac{1}{r} \geq 2$, the possibility $\theta = 0$ is ruled out.
Finally if $\theta = \pi$, then $k = -r$, so

$$(\alpha + \delta)^2 = -r - \frac{1}{r} + 2.$$

But $r \geq 0$, so since $\alpha + \delta$ is real, this implies $r = 1$, so $k = -1$, and T is thus elliptic.