18.112 Functions of a Complex Variable
Fall 2008

Lecture 2: Exponential function & Logarithm for a complex argument

(Replacing Text p.10 - 20)

For $b > 1$, $x \in \mathbb{R}$, we defined in 18.100B,

$$b^x = \sup_{t \in \mathbb{Q},\ t \leq x} b^t$$

(where b^t was easy to define for $t \in \mathbb{Q}$). Then the formula

$$b^{x+y} = b^x b^y$$

was hard to prove directly. We shall obtain another expression for b^x making proof easy.

Let

$$L(x) = \int_1^x \frac{dt}{t}, \quad x > 0.$$

Then

$$L(xy) = L(x) + L(y)$$

and

$$L'(x) = \frac{1}{x} > 0.$$

So $L(x)$ has an inverse $E(x)$ satisfying

$$E(L(x)) = x.$$

By 18.100B,

$$E'(L(x))L'(x) = 1,$$

so

$$E'(L(x)) = x.$$

If $y = L(x)$, so $x = E(y)$, we thus have

$$E'(y) = E(y),$$

It is easy to see $E(0) = 1$, so by uniqueness,

$$E(x) = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} + \cdots \qquad \text{and} \qquad E(1) = e.$$

Theorem 1 $b^x = E(xL(b)), \ \forall x \in \mathbb{R}.$

Proof: Let $u = L(x), \ v = L(y)$, then

$$E(u + v) = E(L(x) + L(y)) = E(L(xy)) = xy = E(u)E(v),$$

$$E(n) = E(1)^n = e^n,$$

and if $t = \frac{n}{m}$,

$$E(t)^m = E(mt) = E(n) = e^n.$$

so

$$E(t) = e^t, \ \ t \in \mathbb{Q}, \ t > 0.$$

Since

$$E(t)E(-t) = 1,$$

So

$$E(t) = e^t, \ \ t \in \mathbb{Q}.$$

Now

$$b^n = E(nL(b))$$

and

$$b^{\frac{1}{m}} = E\left(\frac{1}{m}L(b)\right)$$

since both have same m^{th} power.

$$\left(b^{\frac{1}{m}}\right)^n = b^{\frac{n}{m}} = E\left(\frac{1}{m}L(b)\right)^n = E\left(\frac{n}{m}L(b)\right),$$

so

$$b^t = E(tL(b)), \ t \in \mathbb{Q}.$$

Now for $x \in \mathbb{R}$,

$$b^x = \sup_{t \leq x, \ t \in \mathbb{Q}} (b^t) = \sup_{t \leq x, \ t \in \mathbb{Q}} E(tL(b)) = E(xL(b))$$

since $E(x)$ is continuous. **Q.E.D.**

Corollary 1 *For any* $b > 0, x, y \in \mathbb{R},$ *we have* $b^{x+y} = b^x b^y.$

In particular $e^x = E(x)$, so we have the amazing formula

$$\left(1+1+\frac{1}{2!}+\cdots+\frac{1}{n!}+\cdots\right)^x = 1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}+\cdots.$$

The formula for e^x suggests defining e^z for $z \in \mathbb{C}$ by

$$e^z = 1+z+\frac{z^2}{2!}+\cdots+\frac{z^n}{n!}+\cdots.$$

the convergence being obvious.

Proposition 1 $e^{z+w} = e^z e^w$ *for all* $z, w \in \mathbb{C}$.

Proof: Look at the functions

$$f(t) = e^{tz+w},\ g(t) = e^{tz}e^w$$

for $t \in \mathbb{R}$. Differentiating the series for e^{tz+w} and e^{tz} with respect to t, term-by-term, we see that

$$\frac{df}{dt} = zf(t),\ \frac{dg}{dt} = zg(t)$$

and

$$f(0) = e^w,\ g(0) = e^w.$$

By the uniqueness for these equations, we deduce $f \equiv g$. Thus $f(1) = g(1)$. **Q.E.D.**

Note that if $t \in \mathbb{R}$,

$$e^{it}e^{-it} = 1,\quad \text{and}(e^{it})^{-1} = e^{-it}.$$

Thus

$$|e^{it}| = 1.$$

So e^{it} lies on the unit circle.

Put

$$\cos t = \frac{e^{it}+e^{-it}}{2} = 1-\frac{t^2}{2}+\cdots,$$
$$\sin t = \frac{e^{it}-e^{-it}}{2} = t-\frac{t^3}{3!}+\cdots.$$

Thus we verify the old geometric meaning $e^{it} = \cos t + i\sin t$. Note that the $e^{it}(t \in \mathbb{R})$ fill up the unit circle. In fact by the intermediate value theorem, $\{\cos t \mid t \in \mathbb{R}\}$ fills up the interval $[-1, 1]$, so $e^{it} = \cos t + i \sin t$ is for a suitable t an arbitrary point on the circle.

Note that $z \mapsto e^z$ takes all values $w \in \mathbb{C}$ except 0. For this note

$$e^z = e^x \cdot e^{iy}, \quad z = x + iy.$$

Choose x with

$$e^x = |w|$$

and then y so that

$$e^{iy} = \frac{w}{|w|},$$

then $e^z = w$.

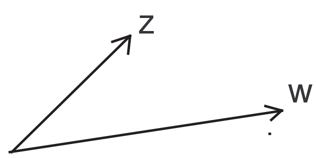

Fig.2-1

If

$$z = |z|e^{i\varphi}, \quad w = |w|e^{i\psi},$$

then

$$\begin{aligned} zw &= |z||w|e^{i(\varphi+\psi)} \\ &= |z||w|(\cos(\varphi+\psi) + i\sin(\varphi+\psi)), \end{aligned}$$

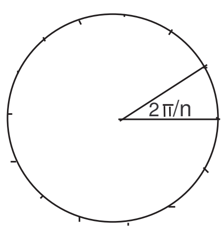

Fig. 2-2

which gives a geometric interpretation of the multiplication.

From this we also have the following very useful formula

$$(\cos\varphi + i\sin\varphi)^n = e^{in\varphi} = \cos n\varphi + i\sin n\varphi.$$

Thus

Theorem 2 *The roots of $z^n = 1$ are $1, \omega, \omega^2, \cdots, \omega^{n-1}$, where*

$$\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}.$$

Geometric meanings for some useful complex number sets:

$$\begin{array}{rcl} |z-a| = r & \longleftrightarrow & \text{circle} \\ |z-a| + |z-b| = r, \ (|a-b| < r) & \longleftrightarrow & \text{ellipse} \\ |z-a| = |z-b| & \longleftrightarrow & \text{perpendicular bisector} \\ \{z \mid z = a + tb, t \in \mathbb{R}\} & \longleftrightarrow & \text{line} \\ \{z \mid \mathrm{Im} z < 0\} & \longleftrightarrow & \text{lower half plane} \\ \{z \mid \mathrm{Im}\left(\dfrac{z-a}{b}\right) < 0\} & \longleftrightarrow & \text{general half plane} \end{array}$$

For x real, $x \mapsto e^x$ has an inverse. This is **NOT** the case for $z \mapsto e^z$, because

$$e^{z+2\pi i} = e^z,$$

thus e^z does not have an inverse. Moreover, for $w \neq 0$,

$$e^z = w$$

has infinitely many solutions:

$$e^x = |w|, \quad e^{iy} = \frac{w}{|w|} \qquad \Longrightarrow \qquad x = \log|w|, \quad y = \arg(w).$$

So

$$\log w = \log|w| + i\arg(w)$$

takes infinitely many values, thus not a function.

Define

$$\mathrm{Arg}(w) \triangleq \text{principal argument of } w \text{ in interval } -\pi < \mathrm{Arg}(w) < \pi$$

and define the principal value of logarithm to be

$$\mathrm{Log}(w) \triangleq \log|w| + i\mathrm{Arg}(w),$$

which is defined in slit plane (removing the negative real axis).

We still have

$$\log z_1 z_2 = \log z_1 + \log z_2$$

in the sense that both sides take the same infinitely many values. We can be more specific:

Theorem 3 *In slit plane,*

$$Log(z_1 z_2) = Log(z_1) + Log(z_2) + n \cdot 2\pi i, \quad n = 0 \text{ or } \pm 1$$

and $n = 0$ if

$$-\pi < Arg(z_1) + Arg(z_2) < \pi.$$

In particular, $n = 0$ if $z_1 > 0$.

Proof: In fact, $\mathrm{Arg}(z_1)$, $\mathrm{Arg}(z_2)$ and $\mathrm{Arg}(z_1 z_2)$ are all in $(-\pi, \pi)$, thus

$$-\pi - \pi - \pi < \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2) - \mathrm{Arg}(z_1 z_2) < \pi + \pi + \pi,$$

but

$$\mathrm{Arg}(z_1) + \mathrm{Arg}(z_2) - \mathrm{Arg}(z_1 z_2) = n \cdot 2\pi i,$$

thus

$$|n| \leq 1.$$

If

$$|\mathrm{Arg}(z_1) + \mathrm{Arg}(z_2)| < \pi,$$

since

$$|\mathrm{Arg}(z_1 z_2)| < \pi,$$

they must agree since difference is a multiple of 2π. **Q.E.D.**