18.112 Functions of a Complex Variable
Fall 2008

Lecture 18: Infinite Products

(Text 191-200)

Remarks on Lecture 18

Problem 1 on p.197: *Suppose that $a_n \to \infty$ (all different, a condition missing in text) and A_n arbitrary complex numbers. Show that there exists an entire function $f(z)$ which satisfies $f(a_n) = A_n$.*

Proof: (A simpler alternative to the hint in text). Let $g(z)$ be an analytic function with simple zeros at the a_n. By the Mittag-Leffler theorem, there exists a meromorphic function h on $\mathbb{C}$ with poles exactly at the points a_n with the corresponding singular part

$$\frac{A_n/b_n}{z-a_n}\,, \quad g(z) = (z-a_n)k(z)\,,\; k(a_n) = b_n \neq 0\,.$$

Then

$$f(z) = g(z)h(z)$$

has the desired property.

Q.E.D.

Remarks on the formula for $\pi \cot \pi z$ (line 8 p.197)

Since the product formula for $\sin \pi z$ has infinitely many factors taking the logarithmic derivative requires justification. Generally, write

$$f(z) = \prod_1^{\infty} f_n(z) = \lim_{N\to\infty} \prod_1^{N} f_n(z) = \lim_{N\to\infty} g_N(z)$$

the convergence being uniform on compacts.
By Theorem 1,

$$f'(z) = \lim_{N\to\infty} g_N'(z)$$

so

$$\frac{f'(z)}{f(z)} = \lim_{N\to\infty} \frac{g_N'(z)}{g_N(z)}.$$

Here $g_N'(z)/g_N(z)$ is given by the rule for differentiating a product.

This remark justifies to proof of (27) as well.

In the text the Gamma function is defined by means of the product formula (29) in §2.4 and the integral formula (42) derived by an interesting residue calculus due to Lindelöf. Here we go a shorter way and derive the product formula from the definition in terms of the integral formula.

The Gamma function can be defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \quad \text{Re} z > 0.$$

Writing

$$f_n(z) = \int_0^n t^{-z-1} e^{-t} dt$$

f_n is holomorphic and

$$|\Gamma(z) - f_n(z)| \leq \left| \int_n^\infty t^{z-1} e^{-t} dt \right| \leq \int_n^\infty t^{\text{Re} z - 1} e^{-t} dt$$

which $\to 0$ uniformly in each half plane $\text{Re} z > \delta$ ($\delta > 0$). Thus $\Gamma(z)$ is holomorphic in $\text{Re} z > 0$. Here are some of its properties
(i) $\Gamma(z+1) = z\Gamma(z)$.
This follows by integration by parts.
(ii) $\Gamma(z)$ extends to a meromorphic function on $\mathbb{C}$ with simple poles at $z = 0, -1, -2, \ldots$.
The function

$$H(z) = \frac{\Gamma(z+1)}{z}$$

is meromorphic in $\text{Re} z > -1$ with a pole at $z = 0$. Since

$$\lim_{z \to 0} z H(z) \neq 0$$

the pole is simple. The residue is $\Gamma(1) = 1$. Also $H(z) = \Gamma(z)$ for $\text{Re} z > 0$. Thus $\Gamma(z)$ is meromorphic in $\text{Re} z > -1$ with simple pole at $z = 0$. Statement (ii) follows by repetition.
(iii) For $x > 0$, $y > 0$

$$\int_0^1 t^{x-1} (1-t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Extend to $\text{Re}\, z > 0$, $\text{Re} w > 0$ $\int_0' t^{z-1} (1-t)^{w-1}\, dt = \frac{\Gamma(z)\Gamma(w)}{\Gamma/2+w}$

Proof:

$$\Gamma(x)\Gamma(y) = \int_0^\infty t^{x-1} e^{-t} dt \int_0^\infty s^{y-1} e^{-s} ds$$

Put $s=tv$. Since integrands are positive, integrals can be interchanged. We get

$$\begin{aligned}\Gamma(x)\Gamma(y) &= \int_0^\infty t^{x-1}e^{-t}dt\int_0^\infty t^y v^{y-1}e^{-tv}dv\\ &= \int_0^\infty v^{y-1}dv\int_0^\infty t^{x+y-1}e^{-(v+1)t}dt \qquad\qquad t=\frac{u}{1+v}\\ &= \int_0^\infty v^{y-1}dv\int_0^\infty u^{x+y-1}e^{-u}(1+v)^{-x-y}du\end{aligned}$$

$$=\Gamma(x+y)\int_0^\infty \frac{v^{y-1}}{(1+v)^{x+y}}dv=\Gamma(x+y)\int_0^1 s^{x-1}(1-s)^{y-1}ds \qquad (1)$$

the last expression coming from $v=s^{-1}(1-s)$. This proves (iii), and it extends to $\operatorname{Re} z>0$, $\operatorname{Re} w>0$.
(iv) $\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin\pi z}$
From (1) we obtain

$$\Gamma(x)\Gamma(1-x)=\int_0^\infty \frac{v^{-x}}{1+v}dv$$

which evaluates to $\pi/\sin(\pi x)$ by the method of Exercise 3(g) p. 161, done in Lecture 15. This proves (iv) by meromorphic continuation.

Since the poles of $\Gamma(z)$ are canceled by zeros of $\sin\pi z$, $\Gamma(1-z)$ is never 0. By (iii) we have for $0<h<\frac{x}{2}$ $\quad z=x+iy$

$$\begin{aligned}\frac{\Gamma(z-h)\Gamma(h)}{\Gamma(z)} &= \int_0^1(1-t)^{z-h-1}t^{h-1}\,dt\\ &= \frac{1}{h}+\int_0^1[(1-t)^{z-h-1}-1]t^{h-1}\,dt\,.\end{aligned}$$

In the integral we use the dominated convergence theorem to justify letting $h\to 0$ under the integral sign. In the interval $[\frac{1}{2},1]$ there is no problem bounding the integrand uniformly for $h<\frac{x}{2}$. On the interval $[0,\frac{1}{2}]$ we have (with $\alpha=z-h-1$)

$$\left|((1-t)^\alpha-1)t^{h-1}\right|\le\left|\frac{(1-t)^\alpha-1}{t}\right|$$

and by l'Hospital's rule this has limit $|\alpha|=|z-h-1|\le|z|+2$ so again the integrand is bounded. Thus we let $h\to 0$ and obtain

$$\frac{\Gamma(z-h)\Gamma(h)}{\Gamma(z)}=\frac{1}{h}+\int_0^1[(1-t)^{z-1}-1]t^{-1}\,dt+o(1)\text{ as }h\to 0\,.$$

The left hand side is using Taylor for $h\to\Gamma(z-h)$ and Laurent for $h\to\Gamma(h)$ both at $h=0$

$$\frac{1}{\Gamma(z)}(\Gamma(z)-h\Gamma'(z)+\cdots)\left\{\frac{1}{h}+A+Bh\cdots\right\}$$

where { } is the Laurent series for $\Gamma(h)$ with center $h = 0$. Equating the constant terms on left hand side and right hand side we get

$$\frac{\Gamma'(z)}{\Gamma(z)} = \int_0^1 (1 - (1-t))^{z-1} t^{-1}\, dt - A \qquad x > 0\,.$$

Writing $t^{-1} = \sum_0^\infty (1-t)^n$ the expression is

$$\int_0^1 \sum_0 [(1-t)^n - (1-t)^{n+z-1}]\, dt - A\,,$$

and since expression in [ ] equals $\frac{(1-t)^n(1-(1-t)^{z-1})}{t} t$ which is bounded by $(1-t)^n Kt$, with integral $K/(n+1)(n+2)$ we can exchange $\int$ and $\sum_n$ by the dominated convergence theorem. Thus our expression equals

$$\begin{aligned} &\sum_0 \int_0^1 [(1-t)^n - (1-t)^{n+z-1}]\, dt - A \\ &= \sum_0^\infty \left(\frac{1}{n+1} - \frac{1}{n+z}\right) - A = 1 - \frac{1}{z} + \sum_1^\infty \left(\frac{1}{n+1} - \frac{1}{n+z}\right) - A \\ &= \sum_1^\infty \left(\frac{1}{n} - \frac{1}{n+1}\right) - \frac{1}{z} + \sum_1^\infty \left(\frac{1}{n+1} - \frac{1}{n+z}\right) - A \\ &= -\frac{1}{z} + \sum_1^\infty \left(\frac{1}{n} - \frac{1}{n+z}\right) - A \end{aligned}$$

so

$$= \frac{\Gamma'(z)}{\Gamma(z)} + \frac{1}{z} = \sum_1^\infty \left(\frac{1}{n} - \frac{1}{n+z}\right) - A\,.$$

Having justified taking logarithmic derivative of an infinite product this gives

$$\frac{1}{\Gamma(z)} = z e^{Cz} \prod_1^\infty \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}} \qquad C = \text{const}\,.$$

Putting $z = 1$ we have

$$1 = e^C \prod_1^\infty \left(1 + \frac{1}{n}\right) e^{-\frac{1}{n}}$$

so

$$1 = e^C \lim_{N\to\infty} \left((N+1) e^{-\left(1 + \frac{1}{2} + \cdots \frac{1}{N}\right)}\right)$$

so

$$0 = C + \lim_{N\to\infty}\left(\log(N+1) - 1 - \frac{1}{2} - \cdots - \frac{1}{N}\right) = C - \gamma$$

so

$$C = \text{the Euler constant } \gamma\,.$$