18.112 Functions of a Complex Variable
Fall 2008

Lecture 17: Mittag-Leffer's Theorem

(Text 187-190)

Theorem 1 (Mittag-Leffer's Theorem) *Let* $\{b_\nu\}$ *be a sequence in* $\mathbb{C}$ *such that*

$$\lim_{\nu\to\infty} b_\nu = \infty,$$

and $P_\nu(\zeta)$ *polynomials without constant term. Then there exist functions* f *meromorphic in* $\mathbb{C}$ *with poles at just the points* b_ν *and corresponding singular parts*

$$P_\nu\left(\frac{1}{z-b_\nu}\right).$$

The most general $f(z)$ *of this kind can be written*

$$f(z) = g(z) + \sum_\nu \left[P_\nu\left(\frac{1}{z-b_\nu}\right) - p_\nu(z)\right] \tag{1}$$

where g *is holomorphic in* z *and the* p_ν *are polynomials.*

Proof: We may assume all $b_\nu \neq 0$. Consider the Taylor series for $P_\nu\left(\frac{1}{z-b_\nu}\right)$ around $z = 0$. It is analytic for $|z| < |b_\nu|$. Let $p_\nu(z)$ be the partial sum up to z^{n_ν} (n_ν to be determined later). Consider the finite Taylor series of

$$\varphi(z) = P_\nu\left(\frac{1}{z-b_\nu}\right)$$

in a disk D with center 0. By (29) on p.126,

$$\varphi_n(z) = \frac{1}{2\pi i}\int_C \frac{\varphi(\zeta)}{\zeta^n(\zeta - z)}d\zeta.$$

Taking C as the circle with center 0 and radius $\dfrac{|b_\nu|}{2}$ and $n = n_\nu + 1$ we deduce

$$|\varphi_{n_\nu+1}(z)| \le \frac{1}{2\pi} 2\pi \frac{|b_\nu|}{2} \frac{M_\nu}{(\frac{1}{2}|b_\nu|)^{n_\nu+1} \cdot \frac{|b_\nu|}{4}} \qquad \text{for } |z| \le \frac{|b_\nu|}{4},$$

where

$$M_\nu = \max_{z \in C} \left| P_\nu \left(\frac{1}{z - b_\nu} \right) \right|.$$

Thus by Theorem 8 on p.125,

$$\left| P_\nu \left(\frac{1}{z - b_\nu} \right) - p_\nu(z) \right| \le 2M_\nu \left(\frac{2|z|}{|b_\nu|} \right)^{n_\nu+1} \qquad \text{for } |z| \le \frac{|b_\nu|}{4}. \tag{2}$$

We now select n_ν large enough so that

$$2^{n_\nu} \ge M_\nu 2^\nu.$$

Then

$$2M_\nu \left(\frac{2|z|}{|b_\nu|} \right)^{n_\nu+1} \le 2^{-\nu} \qquad \text{for } |z| \le \frac{|b_\nu|}{4}.$$

We claim now that the sum (1) converges uniformly in each disk $|z| \le R$ (except at the poles) and thus represents a meromorphic function $h(z)$. To see this we split the sum in (1):

$$h(z) = \sum_{\frac{|b_\nu|}{4} \le R} \left(P_\nu \left(\frac{1}{z - b_\nu} \right) - p_\nu(z) \right) + \sum_{\frac{|b_\nu|}{4} > R} \left(P_\nu \left(\frac{1}{z - b_\nu} \right) - p_\nu(z) \right). \tag{3}$$

Because of (2), the second sum is holomorphic for $|z| \le R$ since $R \le \frac{|b_\nu|}{4}$. The first sum is finite and has

$$P_\nu \left(\frac{1}{z - b_\nu} \right)$$

as the singular part at the pole b_ν.

This proves the existence. If f is any other meromorphic function with these properties, then $f(z) - h(z)$ is holomorphic. **Q.E.D.**

Exercise 3 on p.178

Here we need some preparation on series of the form

$$\sum_{n=1}^{\infty} a_n v_n$$

and use on

$$a_n = (-1)^n, \ v_n = (1+n)^{-s}, \qquad s = \sigma + it.$$

We have if

$$A_n = a_0 + \cdots + a_n,$$

then

$$A_0 v_0 + \sum_{n=1}^{N} (A_n - A_{n-1}) v_n - \sum_{n=0}^{N-1} A_n (v_n - v_{n+1}) = A_N v_N.$$

Lemma 1 *If (A_n) is bounded, $v_n \to 0$, and*

$$\sum_{n=1}^{\infty} |v_n - v_{n+1}| < \infty,$$

then $\sum_{n=0}^{\infty} a_n v_n$ converges.

This is obvious from the identity above.

In our example,

$$v_n = |(1+n)^{-s}| = \frac{1}{(1+n)^{\sigma}},$$

so $v_n \to 0$ even uniformly on compact subsets of $\mathrm{Re}s > 0$. For $v_n - v_{n+1}$ we have

$$v_n - v_{n+1} = \frac{1}{(n+1)^s} - \frac{1}{(n+2)^s} = s \int_{n+1}^{n+2} x^{-s-1} \, dx,$$

so

$$|v_n - v_{n+1}| \leq |s| \frac{1}{(n+1)^{\sigma+1}}.$$

Thus

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^s}$$

converges, and actually uniformly on compact sets in the region $\sigma > 0$ because this is the case with $v_n \to 0$ and $\sum |v_n - v_{n+1}|$.

Exercise 1 on p.186

For a given annulus

$$R_1 < |z-a| < R_2,$$

the expansion

$$\sum_{-\infty}^{\infty} A_n (z-a)^{-n}$$

is unique because the coefficients are determined by (3). For different annuli (even with the same center) the expansion for a given function may be different. Consider

$$\begin{aligned}\frac{1}{z-a} &= \frac{1}{z-b-(a-b)} \\ &= \frac{1}{1-\frac{z-b}{a-b}}\frac{1}{b-a} \\ &= \frac{1}{1-\frac{a-b}{z-b}}\frac{1}{z-b}.\end{aligned}$$

The first formula gives

$$\frac{1}{z-a} = \frac{1}{b-a}\sum_{n=0}^{\infty}\left(\frac{z-b}{a-b}\right)^n \qquad \text{for } 0 < |z-b| < |a-b|,$$

the second

$$\frac{1}{z-a} = \frac{1}{z-b}\sum_{n=0}^{\infty}\left(\frac{a-b}{z-b}\right)^n \qquad \text{for } |a-b| < |z-b| < \infty.$$