18.112 Functions of a Complex Variable
Fall 2008

Lecture 16: Harmonic Functions

(Replacing Text 162-170)

While integrals like $\int_\gamma f(z)\,dz$ and $\int_\gamma M\,dx + N\,dy$ have been defined in the text (p.101), differential forms like dx, dy and $dz = dx + i\,dy$ have not been defined (and the definition is more subtle), we shall develop the theory of harmonic functions (p.162-170) without differential forms.

Definition 1 *A real-valued function $u(z) = u(x, y)$ in a region Ω is* **harmonic** *if it is C^2 and satisfying the equation*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

The Cauchy-Riemann equations for a holomorphic function imply quickly that the real and imaginary parts of a holomorphic function are harmonic. The converse holds if Ω is simply connected:

Theorem 1 *If Ω is simply connected and u harmonic in Ω, there exists a holomorphic function $f(z)$ such that*

$$u(z) = Ref(z).$$

Remark: *Note the condition* <u>*Ω is simply connected*</u> *can not be removed, for example $u(z) = \log|z|$ is harmonic in the punctured plane $\mathbb{C} - \{0\}$, but it cannot be written as real part of a holomorphic function.*

Proof: Put

$$g(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = u_1 + iv_1.$$

Then

$$\frac{\partial u_1}{\partial x} = \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} = \frac{\partial v_1}{\partial y},$$
$$\frac{\partial u_1}{\partial y} = \frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial v_1}{\partial x}.$$

So by the Cauchy-Riemann equation, g is holomorphic. By p.142, since Ω is simply connected,

$$g(z) = f'(z)$$

for some holomorphic function f. Writing

$$f(z) = U(x, y) + iV(x, y),$$

we have by the Cauchy-Riemann equation

$$g(z) = f'(z) = \frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y},$$

so

$$u(x, y) = U(x, y) + \text{constant}.$$

Thus

$$u(z) = \text{Re} f(z) + \text{constant}.$$

Q.E.D.

Corollary 1 (cf. (34) p.134) *If u is harmonic in Ω, then if the disk $|z - z_0| \le r$ lies in Ω,*

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta})\, d\theta.$$

More generally, if the annulus $r_1 \le |z - z_0| \le r_2$ belongs to a region Ω, we have

Theorem 20 *If u is harmonic in Ω, and $\{z : r_1 \le |z - z_0| \le r_2\} \in \Omega$, then*

$$\frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta})\, d\theta = \alpha \log r + \beta, \qquad r_1 \le r \le r_2, \tag{1}$$

where α and β are constants.

Proof: The function $z \mapsto u(z_0 + z)$ is harmonic, so writing the Laplacian in polar coordinates,

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

Denote the left hand side of (1) by $V(r)$, then

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} = 0.$$

Writing this as

$$\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) = 0,$$

the theorem follows. **Q.E.D.**

The Poisson Formula

Let u be harmonic on $|z| \le 1$. Then

$$u = \mathrm{Re}(f)$$

where f is holomorphic on $|z| \le 1$. Consider

$$S(z) = \frac{z+a}{1+\bar{a}z}, \qquad (|a|<1)$$

which maps the unit disk onto itself. Then $f \circ S$ is holomorphic and $u \circ S$ is harmonic (the real part of $f \circ S$). Use the corollary on it with $z_0 = 0$, then

$$u(a) = u(S(0)) = \frac{1}{2\pi}\int_0^{2\pi} u(S(e^{i\varphi}))\ d\varphi.$$

But

$$S(e^{i\varphi}) = \frac{e^{i\varphi}+a}{1+\bar{a}e^{i\varphi}} = e^{i\theta},$$

so

$$e^{i\varphi} = \frac{e^{i\theta}-a}{1-\bar{a}e^{i\theta}}.$$

Hence

$$ie^{i\varphi}\frac{d\varphi}{d\theta} = \frac{ie^{i\theta}-|a|^2 ie^{i\theta}}{(1-\bar{a}e^{i\theta})^2},$$

or

$$\begin{aligned}\frac{d\varphi}{d\theta} &= \frac{ie^{i\theta}-|a|^2 ie^{i\theta}}{(1-\bar{a}e^{i\theta})^2}\cdot\frac{1}{i}\cdot\frac{1-\bar{a}e^{i\theta}}{e^{i\theta}-a} \\ &= \frac{1-|a|^2}{|e^{i\theta}-a|^2}.\end{aligned} \tag{2}$$

This gives

Poisson's Formula ((63) in text)

$$u(a) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\frac{d\varphi}{d\theta}\,d\theta = \frac{1}{2\pi}\int_{|z|=1}\frac{1-|a|^2}{|z-a|^2}u(z)\ d\theta.$$

Schwarz′ Theorem

Theorem 2 (Schwarz′ Theorem) *Let U be a real piecewise continuous function on $|z| = 1$ and define the Poisson integral $u(z) = P_U(z)$ by*

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |a|^2}{|a - e^{i\varphi}|^2} U(e^{i\varphi}) \, d\varphi, \qquad |a| < 1. \tag{3}$$

Then u is harmonic, and

$$\lim_{z \to e^{i\varphi_0}} u(z) = U(e^{i\varphi_0})$$

if U is continuous at $e^{i\varphi_0}$.

Proof: We may assume $\varphi_0 = 0$. Since

$$\frac{1 - |z|^2}{|z - e^{i\varphi}|^2} = \mathrm{Re}\left(\frac{e^{i\varphi} + z}{e^{i\varphi} - z}\right),$$

u is the real part of a holomorphic function, hence harmonic.

Because of (2) formula (3) can be written

$$u(S(0)) = \frac{1}{2\pi} \int_0^{2\pi} U(S(e^{i\varphi})) \, d\varphi.$$

Taking $a = \tanh t$ we obtain as $t \to \infty$

$$\begin{aligned} u(\tanh t) &= \frac{1}{2\pi} \int_0^{2\pi} U\left(\frac{e^{i\varphi} + \tanh t}{\tanh t e^{i\varphi} + 1}\right) \, d\varphi \\ &\longrightarrow \frac{1}{2\pi} \int_0^{2\pi} U(1) \, d\varphi \\ &= U(1). \end{aligned}$$

Q.E.D.

Exercise 5, p.171

Since $\log|1+z|$ is harmonic in $|z|<1$ we have by the mean-value theorem

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|1+re^{i\theta}|\,d\theta=\log 1=0 \tag{4}$$

for $r<1$. We shall now show that

$$\bigl|\log|1+re^{i\theta}|\bigr|$$

is bounded by an integrable function $g(\theta)$. So by the dominated convergence theorem we can let $r\to 1$ under the integral sign, giving the desired result

$$\int_{-\pi}^{\pi}\log|1+e^{i\theta}|\,d\theta=0. \tag{5}$$

Since the integrand $\log|1+e^{i\theta}|$ changes sign on the circle, we split the circle into the two arcs $(-\frac{2\pi}{3},\frac{2\pi}{3})$ and $(\frac{2\pi}{3},\frac{4\pi}{3})$, where we have

$$|1+e^{i\theta}|\geq 1$$

and

$$|1+e^{i\theta}|\leq 1$$

respectively. In the first interval we have $\cos\theta\geq-\frac{1}{2}$ so

$$\frac{\sqrt{3}}{2}\leq|1+re^{i\theta}|\leq|1+e^{i\theta}|=2\cos\frac{\theta}{2},\qquad|\theta|\leq\frac{2\pi}{3},\text{ and } r\geq\frac{1}{2}. \tag{6}$$

In the second interval we put $\theta=\pi+\varphi$ and we see from the geometry, since $|\varphi|\leq\frac{\pi}{3}$, that

$$1\geq|1+re^{i\theta}|=|1-re^{i\varphi}|\geq 1-\cos\varphi=2\cos^2\frac{\theta}{2},\qquad\frac{2\pi}{3}\leq\theta\leq\frac{4\pi}{3}. \tag{7}$$

Since $\log\left|\cos\frac{\theta}{2}\right|$ is integrable, the estimates (6) and (7) show that $|\log|1+re^{i\theta}||$ is bounded by an integrable function $g(\theta)$, so (5) is established.