18.112 Functions of a Complex Variable
Fall 2008

Lecture 14: The Residue Theorem and Application

(Replacing Text 148-154)

Let Ω be a region and $a \in \Omega$. Let $f(z)$ be holomorphic in $\Omega' = \Omega - a$

Definition 1 *The* **residue** *is defined as*

$$R = Res_{z=a} f(z) \triangleq \frac{1}{2\pi i} \int_C f(z)\ dz,$$

where C is any circle contained in Ω with center a.

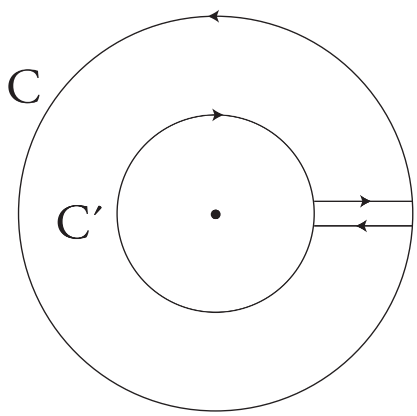

Fig. 14-1

If C' is another circle with center a and

$$C' \subset \Omega,$$

then Cauchy's Theorem for the annulus shows that

$$\mathrm{Res}_{z=a} f(z)$$

is independence of the choice of C.

While the definition can be shown to be equivalent to Definition 3 on p.149 in the text, we shall not need this.

In place of Theorem 17 (Text p.150) we shall prove the following version:

Theorem 17 $'$ *Let f be analytic except for isolated singularities a_j in a region Ω. Let γ be a simple closed curve which has interior contained in Ω and $a_j \notin \gamma$ (all j). Then*

$$\frac{1}{2\pi i} \int_\gamma f(z)\ dz = \sum_j Res_{z=a_j} f(z).$$

where the sum ranges over all a_j inside γ.

Proof:

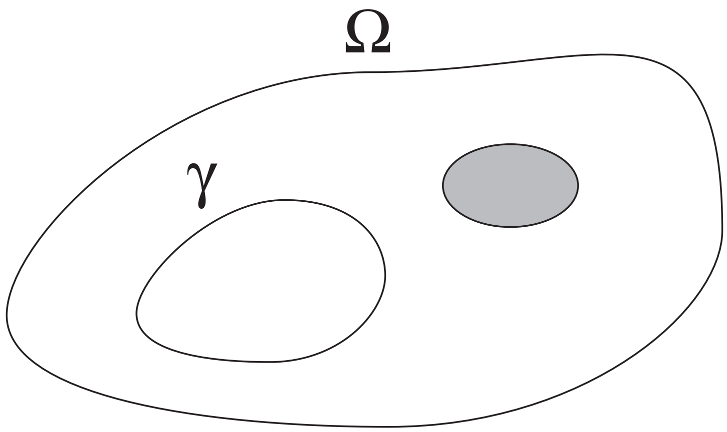

Fig. 14-2

By compactness of γ and its interior, the sum above is finite. For simplicity let a_1, a_2 be the singularities inside γ.

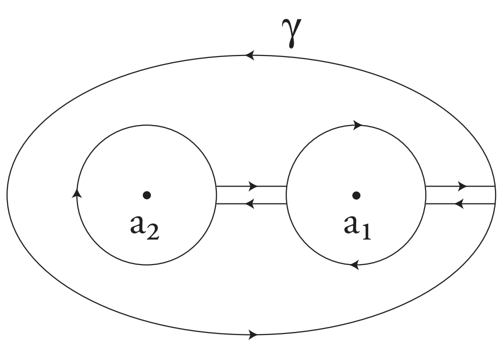

Fig. 14-3

The outside of γ is connected and if we take two disks D_1, D_2 around a_1 and a_2 and connect their boundaries to γ with "bridges" as in Fig. 14-3, the piece remaining in the interior of γ is simply connected (the complement is connected). Thus the integral over the boundary of this region is 0.

Letting the widths of the bridges tend to 0, the theorem follows. **Q.E.D.**

Calculation of residues.

1. *If*

$$\lim_{z\to a} f(z)(z-a)$$

exists and is finite, then it equals $\mathrm{Res}_{z=a} f(z)$.

In fact a is then a pole of $f(z)$, so

$$f(z) = B_h(z-a)^{-h} + \cdots + B_1(z-a)^{-1} + \varphi(z), \qquad B_h \neq 0.$$

Then

$$\frac{1}{2\pi i}\int_C f(z)\,dz = B_1$$

and since the singular part above equals

$$(z-a)^{-h}(B_h + B_{h-1}(z-a) + \cdots + B_1(z-a)^{h-1})$$

the finiteness of the limit implies $h \leq 1$.

2. *If* $f(z) = \frac{g(z)}{h(z)}$ *where* $g(a) \neq 0$ *and* $h(z)$ *has a simple zero at* $z = a$*, then*

$$\mathrm{Res}_{z=a} f(z) = \frac{g(a)}{h'(a)}.$$

In fact

$$\lim_{z\to a} f(a)(z-a) = \lim_{z\to a} g(z)\frac{1}{\frac{h(z)-h(a)}{z-a}} = \frac{g(a)}{h'(a)}.$$

3. *If* f *has a pole of order* h*, then*

$$\mathrm{Res}_{z=a} f(z) = \frac{1}{(h-1)!}\left\{\frac{d^{h-1}}{dz^{h-1}}(z-a)^h f(z)\right\}_{z=a}.$$

In fact

$$f(z) = (z-a)^{-h} g(z),$$

where g is holomorphic at a. So

$$g^{(h-1)}(a) = (h-1)!\frac{1}{2\pi i}\int_C \frac{g(z)}{(z-a)^h}dz = (h-1)!\mathrm{Res}_{z=a} f(z).$$

Example: (from text p.151.)

$$f(z) = \frac{e^z}{(z-a)^2} \qquad \Longrightarrow \qquad \mathrm{Res}_{z=a} f(z) = \left(\frac{d}{dz}e^z\right)_{z=a} = e^a.$$

Application: The Argument Principle.

Theorem 18 $'$ *Let $f(z)$ be meromorphic in Ω, $\gamma \subset \Omega$ a simple closed curve with interior inside Ω. Assume γ passes through no zeros nor poles of f. Then*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = N - P,$$

where N is the number of zeros, P the number of poles inside γ, all counted with multiplicity.

Proof: By theorem 17$'$, the integral is the sum of the residues of $f'(z)/f(z)$.

At a zero a of order h, we have

$$f(z) = (z-a)^h f_h(z), \qquad f_h(a) \neq 0$$

and

$$\frac{f'(z)}{f(z)} = \frac{h}{z-a} + \frac{f_h'(z)}{f_h(z)} \qquad \Longrightarrow \qquad \text{Residue } h,$$

At a pole b of order k, we have similarly

$$\frac{f'(z)}{f(z)} = \frac{-k}{z-b} + \frac{f_h'(z)}{f_h(z)} \qquad \Longrightarrow \qquad \text{Residue } -k.$$

Now the result follows from Theorem 17$'$. **Q.E.D.**

Corollary 1 (Rouche's Theorem) *Let f and g be holomorphic in a region Ω. Let γ be a simple closed curve in Ω with interior $\subset \Omega$. Assume*

$$|f(z) - g(z)| < f(z) \qquad on \quad \gamma.$$

Then f and g have the same number of zeros inside γ, say N_f and N_g.

Proof: (The text does not account into account the case when f and g have common zeros). The inequality implies that f and g are zero-free on γ. Put

$$\psi(z) = \frac{g(z)}{f(z)},$$

then

$$|\psi(z) - 1| < 1$$

on γ, so the curve $\Gamma = \psi(\gamma)$ lies in the disk $|\zeta - 1| < 1$. Hence

$$\frac{1}{2\pi i}\int_\gamma \frac{\psi'(z)}{\psi(z)}dz = \int_\Gamma \frac{d\zeta}{\zeta} = n(\Gamma, 0) = 0$$

(book p.116). Now

$$\begin{aligned}
N_g &= \frac{1}{2\pi i}\int_\gamma \frac{g'(z)}{g(z)}dz \\
&= \frac{1}{2\pi i}\int_\gamma \frac{\psi' f + \psi f'}{\psi f}dz \\
&= \frac{1}{2\pi i}\int_\gamma \frac{\psi'(z)}{\psi(z)}dz + \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz \\
&= N_f.
\end{aligned}$$

This proves the result. **Q.E.D.**

Exercise 2 p.154

We use Rouche's theorem twice, first on $\gamma : |z| = 2$ and then on $\gamma : |z| = 1$.

For $\gamma : |z| = 2$, take $f(z) = z^4,\ g(z) = z^4 - 6z + 3$.

For $\gamma : |z| = 1$, take $f(z) = -6z,\ g(z) = z^4 - 6z + 3$.