18.112 Functions of a Complex Variable
Fall 2008

Lecture 13: The General Cauchy Theorem

(Replacing Text 137-148)

Here we shall give a brief proof of the general form of Cauchy's Theorem. (cf: John D. Dixon, A brief proof of Cauchy's integral theorem, *Proc. Amer. Math. Soc.* 29, (1971) 625-626.)

Definition 1 *A closed curve γ in an open set Ω is* **homologous to** 0 *(written $\gamma \sim 0$) with respect to Ω if*

$$n(\gamma, a) = 0 \qquad \textit{for all} \;\; a \notin \Omega.$$

Definition 2 *A region is* **simply connected** *if its complement with respect to the extended plane is connected.*

Remark: If Ω is simply connected and $\gamma \subset \Omega$ a closed curve, then $\gamma \sim 0$ with respect to Ω. In fact, $n(\gamma, z)$ is constant in each component of $\mathbb{C} - \gamma$, hence constant in $\mathbb{C} - \Omega$ and is 0 for z sufficiently large.

Theorem 1 (Cauchy's Theorem) *If f is analytic in an open set Ω, then*

$$\int_\gamma f(z)\; dz = 0$$

for every closed curve $\gamma \subset \Omega$ such that $\gamma \sim 0$.

In particular, if Ω is simply connected then $\int_\gamma f(z)\, dz = 0$ for every closed $\gamma \subset \Omega$.

We shall first prove

Theorem 2 (Cauchy's Integral Formula) *Let f be holomorphic in an open set Ω. Then*

$$n(\gamma, z) f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta \tag{1}$$

where $\gamma \sim 0$ with respect to Ω.

Proof: The prove is based on the following three claims.

Define $g(z,\zeta)$ on $\Omega \times \Omega$ by

$$g(z,\zeta) = \begin{cases} \frac{f(\zeta)-f(z)}{\zeta - z} & \text{for} \quad z \neq \zeta, \\ f'(z) & \text{for} \quad z = \zeta. \end{cases}$$

Claim 1: *g is continuous on $\Omega \times \Omega$ and holomorphic in each variable and $g(z,\zeta) = g(\zeta, z)$.*

Clearly g is continuous outside the diagonal in $\Omega \times \Omega$. Let (z_0, z_0) be a point on the diagonal and $D \subset \Omega$ a disk with center z_0. Let $z \neq \zeta$ in D. Then by Theorem 8

$$g(z,\zeta) - g(z_0, z_0) = f'(\zeta) + \frac{1}{2} f_2(z)(z-\zeta) - f'(z_0).$$

So the continuous at (z_0, z_0) is obvious.

For the holomorphy statement, it is clear that for each $\zeta_0 \in \Omega$ the function

$$z \mapsto g(z, \zeta_0)$$

is holomorphic on $\Omega - \zeta_0$. Since

$$\lim_{z \to \zeta_0} g(z,\zeta_0)(z - \zeta_0) = 0$$

the point ζ_0 is a removable singularity (Theorem 7, p.124), so

$$z \mapsto g(z, \zeta_0)$$

is indeed holomorphic on Ω. This proves Claim 1.

Let

$$\Omega' = \{z \in \mathbb{C} - (\gamma) \ : \ n(\gamma, z) = 0\}.$$

Define function h on $\mathbb{C}$ by

$$h(z) = \frac{1}{2\pi i}\int_\gamma g(z,\zeta)\ d\zeta, \qquad z \in \Omega; \tag{2}$$

$$h(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\ d\zeta, \qquad z \in \Omega'. \tag{3}$$

Since both expression agree on $\Omega \cap \Omega'$ and since $\Omega \cup \Omega' = \mathbb{C}$, this is a valid definition.

Claim 2: *h is holomorphic.*

This is obvious on the open sets Ω' and $\Omega - \gamma$. To show holomorphy at $z_0 \in \gamma$, consider a disk $D \subset \Omega$ with center z_0. Let δ be any closed curve in D. Then

$$\begin{aligned}\int_\delta h(z)\ dz &= \frac{1}{2\pi i}\int_\delta \left(\int_\gamma g(z,\zeta)\ d\zeta\right) dz \\ &= \frac{1}{2\pi i}\int_\gamma \left(\int_\delta g(z,\zeta)\ dz\right) d\zeta.\end{aligned}$$

For each ζ,

$$z \mapsto g(z,\zeta)$$

is holomorphic on D (even Ω). So by the Cauchy's theorem for disks,

$$\int_\delta g(z,\zeta)\ dz = 0.$$

Now the Morera's Theorem implies h is holomorphic.

Now we can prove:

Claim 3: *$h \equiv 0$, so (1) holds.*

We have $z \in \Omega'$ for $|z|$ sufficiently large. So by (3),

$$\lim_{z\to\infty} h(z) = 0.$$

By Liouville's Theorem, $h \equiv 0$. **Q.E.D.**

Proof of Theorem 1: To derive Cauchy's theorem, let $z_0 \in \Omega - \gamma$ and put

$$F(z) = (z - z_0)f(z).$$

By (1),

$$\begin{aligned}\frac{1}{2\pi i}\int_\gamma f(z)\, dz &= \frac{1}{2\pi i}\int_\gamma \frac{F(z)}{z - z_0}dz \\ &= n(\gamma, z_0)F(z_0) \\ &= 0.\end{aligned}$$

Q.E.D.

Note finally that Corollary 2 on p.142 is an immediately consequence of Cauchy's Theorem.