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18.112 Functions of a Complex Variable  
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# Lecture 12: The Local Mapping. Schwarz's Lemma and non-Euclidean interpretation

(Text 130-136)

## Remarks on Lecture 12

### Proof of Theorem 11, p.131.

The last formula on p.131 reads

$$\sum_j n(\gamma, z_j(a)) = \sum_j n(\gamma, z_j(b)) \quad (1)$$

provided  $a$  and  $b$  belong to the same component of the set  $\mathbb{C} - \Gamma$ . In (1) we take for  $\gamma$  the circle

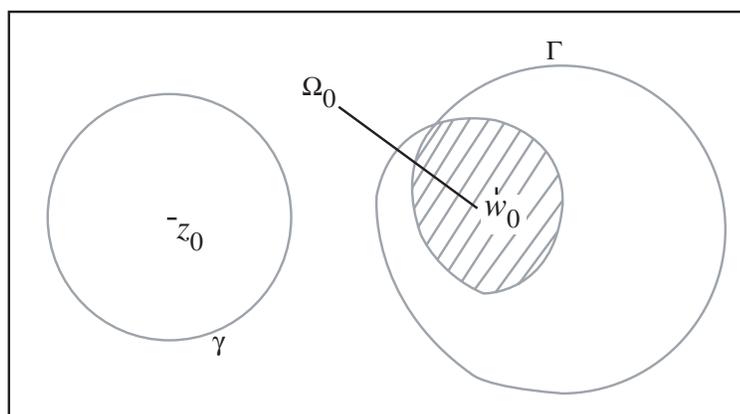
$$z(t) = z_0 + \epsilon e^{it} \quad (0 \leq t \leq 2\pi),$$

where  $\epsilon > 0$  is so small that  $z_0$  is the only zero of

$$f(z) - w_0 = 0$$

inside  $\gamma$ . (The zeroes of  $f(z) - w_0 = 0$  are isolated.)

As before let  $\Gamma = f(\gamma)$  and let  $\Omega_0$  denote the component of  $\mathbb{C} - \Gamma$  containing  $w_0$ .



Since  $\gamma$  is a circle,

$$n(\gamma, z_j(a)) = 0 \text{ or } 1$$

depending on whether  $z_j(a)$  is outside  $\gamma$  or inside  $\gamma$  (Since  $a \notin \Gamma$ ,  $z_j(a) \notin \gamma$ ). Thus the left hand side of (1) equals the number of points inside  $\gamma$  where  $f$  takes the value  $a$ .

In (1) we now take  $a$  arbitrary in  $\Omega_0$  and take  $b = w_0$ . By the choice of  $\gamma$  the right hand side of (1) equals  $n$ . Equation (1) therefore shows that each value in  $\Omega_0$  is taken  $n$  times by  $f$  inside  $\gamma$  (multiple roots counted according to their multiplicity). In particular, this holds for any disk  $|w - w_0| < \delta$  inside  $\Omega_0$  with center  $w_0$ . **Q.E.D.**

**Remark:** In dealing with winding numbers  $n(\delta, z)$ , we have to pay attention to the parametrization. Thus if

$$\gamma(t) = e^{it} \quad (0 \leq t \leq 2\pi), \quad f(z) = z^n$$

and  $\Gamma = f(\gamma)$ , we have

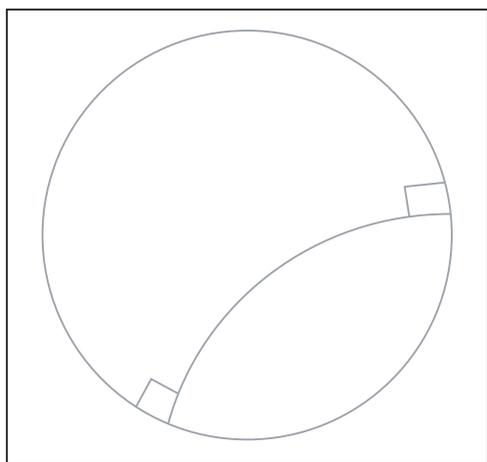
$$n(\gamma, 0) = 1, \quad n(\Gamma, 0) = n$$

although  $\gamma$  and  $\Gamma$  are represented geometrically by the same point set.

## Non-Euclidean Plane

This is the unit disk  $|z| < 1$  with the following convention:

- a) Non-Euclidean point = Point in disk;
- b) Non-Euclidean line = Arc in disk perpendicular to the boundary.

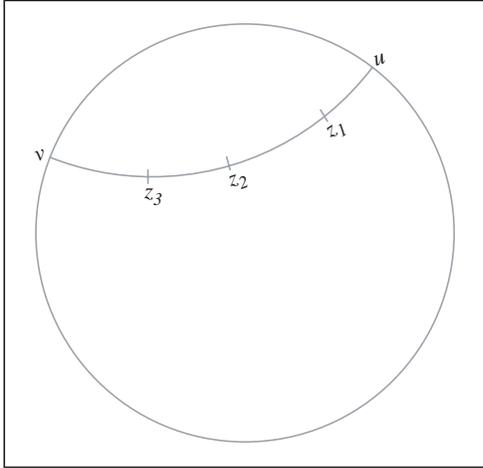


This model satisfies all Euclid's axioms except the famous **Parallel Axiom**:

Given a point  $p$  outside a line  $l$ , there is exactly one line through the point which does not intersect  $l$ .

This axiom clearly fails in the above model; thereby solving the 2000 year old problem of proving the Parallel Axiom on the basis of the other axioms. It cannot be done!

We can now introduce distance in the non-Euclidean plane  $D$ .



Given  $z_1, z_2 \in D$ , mark the points  $u, v$  of intersection with  $|z| = 1$ , in the order indicated. We put

$$\begin{aligned} d(z_1, z_2) &= \frac{1}{2} \log (z_1, z_2, v, u) \\ &= \frac{1}{2} \log \left( \frac{z_1 - v}{z_1 - u} : \frac{z_2 - v}{z_2 - u} \right). \end{aligned}$$

The cross ratio is real (page 79) and the geometry shows easily that the cross ratio is  $\geq 1$ . So

$$d(z_1, z_2) \geq 0,$$

$$d(z_1, z_2) = d(z_2, z_1).$$

It is also easy to show from the formula that

$$d(z_1, z_3) = d(z_1, z_2) + d(z_3, z_2).$$

Consider a fractional linear transformation

$$z \longrightarrow e^{i\varphi} \frac{z - z_2}{1 - \bar{z}_2 z}$$

mapping

$$z_2 \longrightarrow 0 \quad \text{and} \quad z_1 \longrightarrow \left| \frac{z_1 - z_2}{1 - \bar{z}_2 z_1} \right|.$$

Then by the order of the points  $v, z_2, z_1, u$  we see that

$$v \longrightarrow -1 \quad \text{and} \quad u \longrightarrow 1.$$

The invariance of the cross ratio gives

$$\begin{aligned} d(z_2, z_1) &= \frac{1}{2} \log \left( 0, \left| \frac{z_2 - z_1}{1 - \bar{z}_2 z_1} \right|, -1, 1 \right) \\ &= \frac{1}{2} \log \left( \frac{|1 - \bar{z}_2 z_1| + |z_1 - z_2|}{|1 - \bar{z}_2 z_1| - |z_1 - z_2|} \right) \quad (\text{Exercise 7}). \end{aligned}$$

Thus

$$d(z, z + \Delta z) = \frac{1}{2} \log \left( 1 + \frac{2|\Delta z|}{|1 - \bar{z}(z + \Delta z)| - |\Delta z|} \right).$$

So since

$$\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$$

we deduce

$$\lim_{\Delta z \rightarrow 0} \frac{d(z, z + \Delta z)}{|\Delta z|} = \frac{1}{1 - |z|^2}.$$

This suggests defining a non-Euclidean length of a curve

$$\gamma : z(t) \quad (\alpha \leq t \leq \beta)$$

by

$$L(\gamma) = \int_{\gamma} \frac{|dz|}{1 - |z|^2}.$$

Exercises 1 and 6 now give the following geometric interpretation of Schwarz's Lemma:

Ex 1. Formula (30) implies by division, letting  $z \rightarrow z_0$ ,

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}.$$

Ex 6. Let  $f : D \rightarrow D$  be holomorphic and  $f(\gamma)$  the image curve

$$w(t) = f(z(t)) \quad \alpha \leq t \leq \beta$$

of the curve  $\gamma$  above. Then

$$\begin{aligned} L(f(\gamma)) &= \int_{\alpha}^{\beta} \frac{|w'(t)|}{1 - |w(t)|^2} dt \\ &= \int_{\alpha}^{\beta} \frac{|f'(z(t))z'(t)|}{1 - |f(z(t))|^2} dt \\ &\leq \int_{\alpha}^{\beta} \frac{|z'(t)|}{1 - |z(t)|^2} dt \quad (\text{by Ex 1.}) \\ &= L(\gamma). \end{aligned}$$

Thus

$$L(f(\gamma)) \leq L(\gamma)$$

as stated.