18.112 Functions of a Complex Variable
Fall 2008

Lecture 11: Isolated Singularities

(Text 126-130)

Remarks on Lecture 11

Singularities: Let $f(z)$ be holomorphic in a disk $0 < |z-a| < \delta$ with the center a removed.

(i) If

$$\lim_{z \to a} f(z)$$

exist or if just

$$\lim_{z \to a} f(z)(z-a) = 0,$$

then a is a removable singularity and f extends to a holomorphic function on the whole disk $|z-a| < \delta$.

(ii) If

$$\lim_{z \to a} f(z) = \infty,$$

a is said to be a pole. In this case

$$f(z) = (z-a)^{-h} f_h(z),$$

where h is a positive integer and $f_h(z)$ is holomorphic at a and $f_h(a) \neq 0$. We also have the polar development

$$f(z) = B_h(z-a)^{-h} + \cdots + B_1(z-a)^{-1} + \varphi(z),$$

where $\varphi(z)$ is holomorphic at a.

If neither (i) nor (ii) holds, a is said to be an essential singularity.

Theorem 9 *A holomorphic function comes arbitrarily close to any complex value in every neighborhood of an essential singularity.*

Simplified Proof: Suppose statement false. Then $\exists A \in \mathbb{C}$ and $\delta > 0$ and $\epsilon > 0$ such that

$$|f(z) - A| > \delta \qquad \text{for } |z - a| < \epsilon.$$

Then

$$\lim_{z \to a} (z-a)^{-1}(f(z) - A) = \infty.$$

So

$$(z-a)^{-1}(f(z) - A)$$

has a pole at $z = a$. Thus

$$f(z) - A = (z-a)(z-a)^{-h} g(z),$$

where $h \in \mathbb{Z}^+$ and $g(z)$ is holomorphic at $z = a$.

If $h = 1$, $f(z)$ has a removable singularity at $z = a$. If $h > 1$, $f(z) - A$ has a pole at $z = a$ and so does $f(z)$. Both possibilities are excluded by assumption, so the proof is complete. **Q.E.D.**

Exercise 4 on p.130.

Suppose f is meromorphic in $\mathbb{C} \cup \{\infty\}$. We shall prove f is a rational function. If ∞ is a pole, we work with $g = 1/f$, so we may assume ∞ is not a pole. It is not an essential singularity, so ∞ is a removable singularity. Thus for some $R > 0$, $f(z)$ is bounded for $|z| \geq R$. Since the poles of $f(z)$ are isolated, there are just finitely many poles in the disk $|z| \leq R$. (Poles of $f(z)$ are zeroes of 1/f(z).) At a pole a, use the polar development near a

$$f(z) = B_h(z-a)^{-h} + \cdots + B_1(z-a)^{-1} + \varphi(z).$$

The equation shows that φ extends to a meromorphic function on $\mathbb{C} \cup \infty$ with one less pole than $f(z)$. We can then do this argument with $\varphi(z)$ and after iteration we obtain

$$f(z) = \sum_{i=1}^{n} P_i\left(\frac{1}{z - a_i}\right) + g(z),$$

where P_i are polynomials and g is holomorphic in $\mathbb{C}$. The formula shows that g is bounded for $|z| > geR$ and being analytic on $|z| \leq R$, it thus must be bounded on $\mathbb{C}$. By Liouville's theorem, it is constant. So f is a rational function.