18.112 Functions of a Complex Variable
Fall 2008

Lecture 10: The Special Cauchy's Formula and Applications

(Text 118-126)

Remarks on Lecture 10

Exercise 6 on page 108

The values of $f(z)$ lie in the disk $|w-1| < 1$ which is contained in the slit plane where $\text{Log} w$ is defined. thus $\text{Log} f(z)$ is well-defined and holomorphic in Ω and has derivative

$$\frac{1}{f(z)} f'(z).$$

Thus

$$\int_\gamma \frac{f'(z)}{f(z)} dz = 0$$

by the Primitive theorem.

Exercise 2 on page 120

By using the substitution $w = \varphi(z) = -z$ we have

$$\int_{\varphi(\gamma)} \frac{dw}{w^2+1} = \int_\gamma \frac{-dz}{z^2+1}.$$

Since $\varphi(\gamma) = \gamma$ (including the orientation). Thus the integral is 0.

Also

$$\frac{1}{z^2+1} = \frac{1}{z-i} - \frac{1}{z+i}$$

and

$$n(\gamma, i) = n(\gamma, -i),$$

so again the total integral is 0.

Exercise 3 on page 120

On $|z| = \rho$, we can write $z = \rho e^{i\theta}$, thus

$$\frac{dz}{d\theta} = \rho e^{i\theta} i,$$

so

$$\frac{dz}{z} = i \; d\theta,$$

and

$$|dz| = \rho d\theta = -i\rho \frac{dz}{z}.$$

Thus

$$\begin{aligned} \int_{|z|=\rho} \frac{|dz|}{|z-a|^2} &= -i\rho \int_{|z|=\rho} \frac{dz}{z(z-a)(\frac{\rho^2}{z} - \bar{a})} \\ &= -i\rho \left[\frac{1}{\rho^2 - |a|^2} \int_{|z|=\rho} \frac{dz}{z-a} + \frac{\bar{a}}{\rho^2 - |a|^2} \int_{|z|=\rho} \frac{dz}{\rho^2 - \bar{a}z} \right]. \end{aligned}$$

If $|a| > \rho$, the first term is 0, the other term is

$$\frac{1}{\bar{a}} \int_{|z|=\rho} \frac{dz}{\frac{\rho^2}{\bar{a}} - z} = -2\pi i \frac{1}{\bar{a}},$$

so the result is

$$\frac{2\pi\rho}{|a|^2 - \rho^2}.$$

If $|a| < \rho$, then the second is 0 and the other is

$$-i\rho \; 2\pi i \frac{1}{\rho^2 - |a|^2} = \frac{2\pi\rho}{\rho^2 - |a|^2}.$$

Thus in both cases the result is

$$\left| \frac{2\pi\rho}{\rho^2 - |a|^2} \right|.$$

▶ The Taylor's Theorem (with remainder) proved in pp.125-126 should be stated as follows:

Theorem 1 (Taylor's Theorem) *If $f(z)$ is analytic in a region Ω containing a, one has*

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \cdots + \frac{f^{n-1}(a)}{(n-1)!}(z-a)^{n-1} + f_n(z)(z-a)^n,$$

where $f_n(z)$ is analytic in Ω. Moreover, if C is the boundary of a closed disk contained in Ω with center a, then $f_n(z)$ has the representation

$$f_n(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)\,d\zeta}{(\zeta-a)^n(\zeta-z)} \qquad (z \text{ inside } C).$$