18.112 Functions of a Complex Variable
Fall 2008

Lecture 1: The algebra of Complex numbers

(Text 1-11 & 19-20)

Remarks on Lecture 1

► On p.19-20, it is stated that each circle in $\mathbb{C}$

$$(x-a)^2 + (y-b)^2 = r^2 \tag{1}$$

has the form

$$(\alpha_0 - \alpha_3)(x^2 + y^2) - 2\alpha_1 x - 2\alpha_2 y + \alpha_0 + \alpha_3 = 0,$$

so the mapping $z \mapsto Z$ maps circles in the plane to circles on S. Solving the equations

$$a = \frac{\alpha_1}{\alpha_0 - \alpha_3},\ b = \frac{\alpha_2}{\alpha_0 - \alpha_3},\ r^2 - a^2 - b^2 = -\frac{\alpha_0 + \alpha_3}{\alpha_0 - \alpha_3}$$

for $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ is disagreeable so we instead determine the image of the curve (1) under the map $z \mapsto Z$. Using the formulas (24)-(26) and

$$1 - x_3 = \frac{2}{1 + |z|^2},$$

formula (1) becomes

$$ax_1 + bx_2 + \frac{1 + r^2 - a^2 - b^2}{2} x_3 = \frac{a^2 + b^2 - r^2 + 1}{2}.$$

This is a plane which must intersect the sphere so has distance < 1 from 0.

▶ The formula (28) can be proved geometrically as follows (Exercise 4):

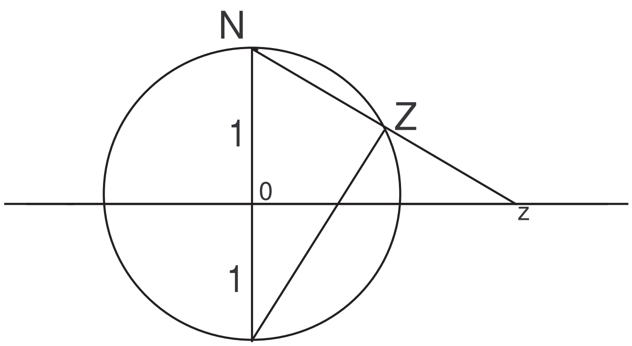

Fig. 1-1

Let $Z \in S$ lie on the plane

$$x_2 = 0.$$

The angles at Z are right angles, so by similar triangles:

$$\frac{d(N,Z)}{2} = \frac{1}{d(N,z)} = \frac{1}{\sqrt{1+|z|^2}}.$$

Thus

$$\frac{d(N,Z)}{d(N,z')} = \frac{2}{\sqrt{1+|z|^2}\sqrt{1+|z'|^2}}$$

and by symmetry this is

$$\frac{d(N,Z')}{d(N,z)}.$$

Thus the triangles $\triangle NZZ'$ and $\triangle Nzz'$ are similar, so the above ratio is

$$\frac{d(Z,Z')}{|z-z'|}.$$

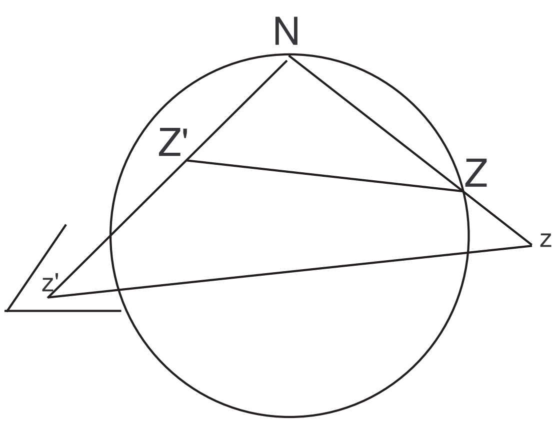

Fig. 1-2

This proves (28).

▶ Finally we show that the spherical representation $z \mapsto Z$ is conformal. This means that if l and m are two lines in the plane intersecting in z at an angle α, then the corresponding circles C and D through N and Z intersect Z at the same angle α. Consider the tangent plane π to S at the point N. the plane through Z and l intersects π in a line l'. Similarly the plane through Z and m intersect π in m'. Clearly l' and m' intersect at N at the same angle α. Since they are tangents to C and D at N, C and D must intersect at the angle α both at N and at Z.