18.112 Functions of a Complex Variable
Fall 2008

Solution for 18.112 Mid 2

Problem 1.
Solution: The function

$$f(z) = \frac{1}{e^z - 1}$$

is analytic in $\mathbb{C} - \{2n\pi i, n \in \mathbb{Z}\}$, and has simple pole at points $z = 2n\pi i$. Thus there are three poles in the region bounded by γ, which correspond to $n = 0, \pm 1$. Moreover, at each pole z, the residue equals to

$$\frac{1}{(e^z - 1)'} = \frac{1}{e^z} = 1.$$

By residue theorem,

$$\int_\gamma \frac{1}{e^z - 1} dz = 2\pi i(1 + 1 + 1) = 6\pi i.$$

Problem 2.
Solution: Let

$$u(z) = \text{Re} f(z),$$

then by formula (66), there exists constant C such that for any $|z| < R$,

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta|=R} \frac{\zeta + z}{\zeta - z} u(\zeta) \frac{d\zeta}{\zeta} + iC.$$

Thus

$$f'(z) = \frac{1}{\pi i} \int_{|\zeta|=R} \frac{u(\zeta)}{(\zeta - z)^2} d\zeta$$

for any $|z| < R$.

Now for any z, take $R > 2|z|$ large enough such that

$$\frac{|u(\zeta)|}{|\zeta|} < 1$$

for any $|\zeta| \geq R$. Then

$$|f'(z)| \leq \frac{1}{\pi} \int_{|\zeta|=R} \frac{|\zeta|}{|\zeta - z|^2} |d\zeta| \leq \frac{1}{\pi} \cdot 2\pi R \cdot \frac{R}{(R/2)^2} = 8.$$

Thus $f'(z)$ is bounded analytic function on $\mathbb{C}$. By Liouville's theorem, $f'(z)$ is constant, so $f(z) = az + b$ is linear. By condition

$$\frac{u(z)}{z} \to 0,$$

we see that $a = 0$, which implies that f is a constant.

N.B. You can also prove that

$$\frac{\mathrm{Im} f(z)}{z} \to 0,$$

thus

$$\frac{f(z)}{z} \to 0 \text{ as } z \to \infty.$$

So by *Problem 4* or *Problem 5* in **Mid 1**, f is a polynomial, and thus f is a constant.

Problem 3.

Solution: By Cauchy's formula,

$$\begin{aligned}
|f^{(n)}(0)| &\leq \frac{n!}{2\pi} \int_{|\zeta|=r} \frac{|f(\zeta)|}{|\zeta^{n+1}|} |d\zeta| \\
&\leq \frac{n!}{2\pi} \cdot 2\pi r \cdot \frac{1}{1-r} \frac{1}{r^{n+1}} \\
&= \frac{n!}{(1-r)r^n}
\end{aligned}$$

for $0 < r < 1$. On the other hand,

$$\begin{aligned}
\frac{1}{(1-r)r^n} &= \frac{1}{n^n} \frac{1}{(1-r)(r/n)^n} \\
&\geq \frac{1}{n^n} \left(\frac{n+1}{1 - r + \frac{r}{n} + \cdots + \frac{r}{n}} \right)^{n+1} \\
&= \frac{(n+1)^{n+1}}{n^n}
\end{aligned}$$

by Algebraic-Geometric Mean Value inequality, with equality holds if and only if

$$1 - r = \frac{r}{n},$$

i.e.

$$r = \frac{n}{n+1}.$$

So the best estimate of $f^{(n)}(0)$ that Cauchy's formula will yield is

$$|f^{(n)}(0)| \leq \frac{n!(n+1)^{(n+1)}}{n^n} = (n+1)!(1+\frac{1}{n})^n.$$

N.B. You can also get the minimal value of

$$\frac{1}{(1-r)r^n}$$

by computing derivatives.

Problem 4.

Solution: Let

$$\begin{aligned} f(z) &= z^7 - 2z^5 + 6z^3 - z + 1, \\ g(z) &= 6z^3, \\ h(z) &= z^7. \end{aligned}$$

Then we have

$$|f(z) - g(z)| = |z^7 - 2z^5 - z + 1| \leq 5 < 6 = |g(z)|$$

on the curve $|z| = 1$, and

$$|f(z) - h(z)| = |-2z^5 + 6z^3 - z + 1| \leq 115 < 128 = |h(z)|$$

on the curve $|z| = 2$. Thus by Rouche's theorem, f has 3 roots (as g) in $|z| < 1$ and 7 roots (as h) in $|z| < 2$.

N.B. You can also choose $g(z)$ to be $6z^3 + 1$ or $6z^3 - z$, or choose $h(z) = z^7 + 1$ etc.