18.112 Functions of a Complex Variable
Fall 2008

Problems for 18.112 Final Examination
(Open Book)

Dec. 20, 2006

1. (20') Let a, b, c be complex numbers satisfying

$$\frac{b-a}{c-a} = \frac{a-c}{b-c}.$$

Considering the triangle with vertices a, b, c. Prove

$$|b-a| = |c-a| = |b-c|.$$

2. (15') Find where the series

$$\sum_{n=1}^{\infty} \frac{z^n}{1+z^{2n}}$$

converges and determine where the sum $f(z)$ is holomorphic. Give reasons for your answer.

3. (15') Evaluate

$$\int_{\gamma} \frac{|z|e^z}{z^2} dz$$

where γ is the circle with radius 2 and center 0.

4. (15') Prove that if $f(z)$ has a pole of order h at z_0, then

$$\text{Res}_{z=z_0} f(z) = \frac{1}{(h-1)!} \left\{ \frac{d^{h-1}}{dz^{h-1}} (z-z_0)^h f(z) \right\}_{z=z_0}.$$

5. (20') Using the geometric series, find Laurent expansions for

$$f(z) = \frac{1}{(z-1)(z-2)}$$

valid in $|z| < 1$ and valid in $|z| > 2$.

6. (15') Let $f(z)$ be analytic in $|z| \leq 1$. Suppose that $|f(z)| < 1$ if $|z| = 1$. Show that the equation

$$f(z) - z = 0$$

has exactly one solution inside $|z| = 1$.